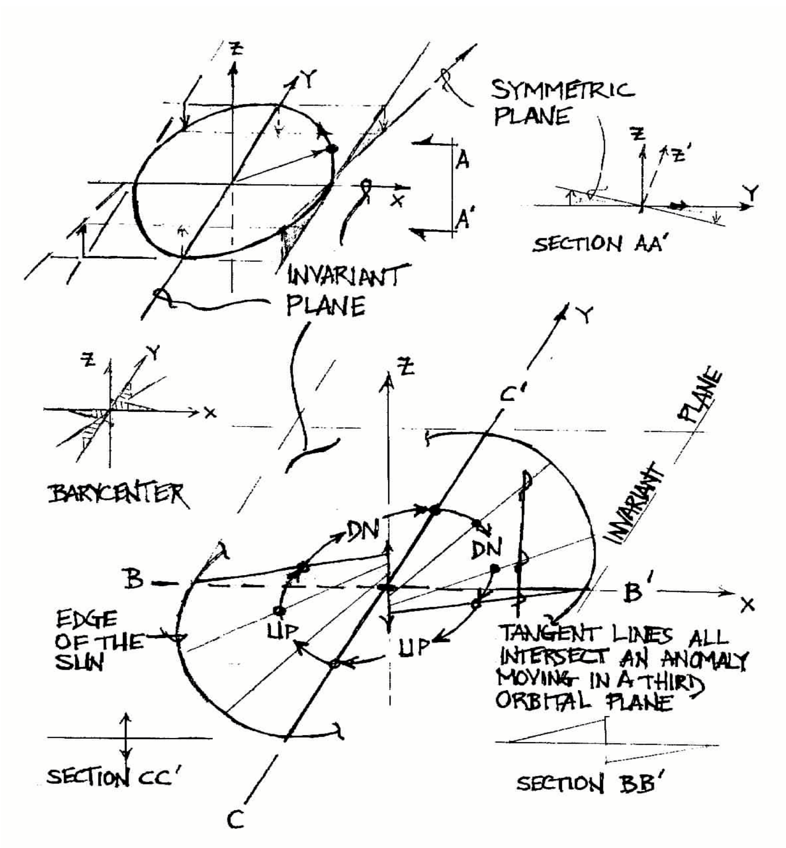
A Geometric Derivation of

Relativity

From the Two Body Problem

by WH Clark



i. Table of Contents

ii. Introduction

In the classical era of science, Newton's Laws of Motion were derived graphically; as were Kepler's Laws describing the orbits of the planets; and all the fundamental relationships in the early Theory of Orbits had a graphical basis. They were all subsequently proven using mathematics, typically the calculus, but the original theory was built using simple geometry.

This book follows the early days in the renaissance of science, from Copernicus to Kepler to Newton, Poincare and Euler. The whole body of work on the Two Body Problem is presented, with both graphical and rigorous mathematical proofs. In many cases the two cannot be separated because the graphical proofs are integral to the mathematical ones.

Sometimes the geometric analysis is more exact than the calculus. There are assumptions made in many derivations that are not strictly valid, or which lead to some interesting questions. Back then I'm sure they said "It's all relative," and actually it is… these miniscule flaws in fundamental theories have no consequence in Newtonian mechanics; but at the Relativistic level where everything is infinitely faster, tiny misrepresentations blow up into observable phenomena.

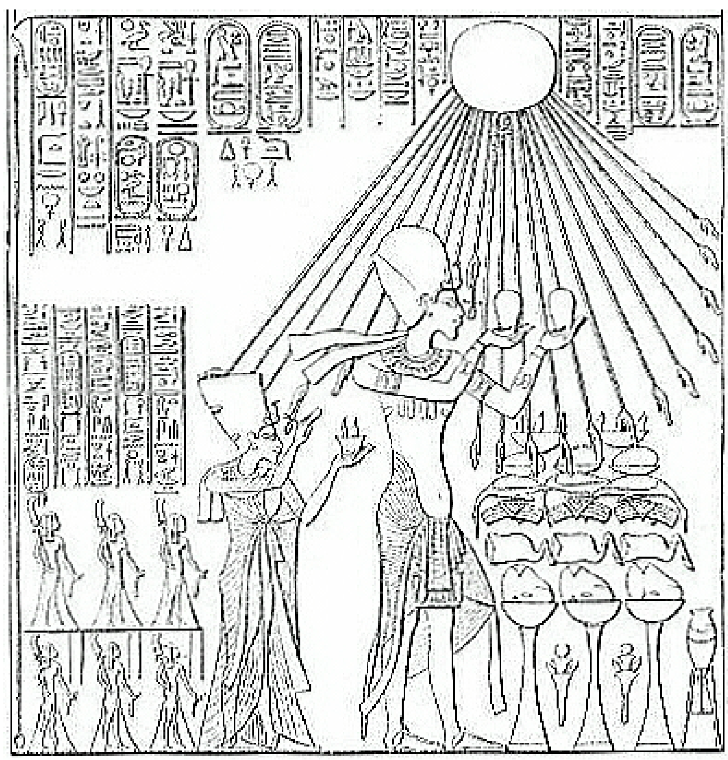
The text draws a consistent theme through the Two Body Theory and little anomalies along the way, slowly building the basis for what in the modern day is a strange idea: that Relativity has to have a geometric basis, like virtually every proof in astronomy that preceded it. The last two sections present speculative papers outlining just such a proof.

Along the way many intriguing insights are offered into fundamental mathematical ideas, like matrix algebra, complex analysis, Fourier Series, and Fourier and Laplace Transforms. Perhaps the most rewarding part of the exposition is how these abstract concepts are given a simple, solid, geometric structure. These insights are so compelling that the whole Relativity thing becomes quite the anti climax.

WH Clark
Austin, Texas
10.29.2005

iii. Dedication

This book is dedicated to the priests of ancient Egypt who build the foundations of astronomy over the course of three thousand years of careful observation and thought. It doesn't matter that the Greeks and Babylonians; Romans and Arabs all stole their ideas and claimed them as their own. It was the Egyptians who first observed the heavens and who identified most objects in the night sky for what they are ~ stars, planets, and comets. It was the Egyptians who systemized the analysis of the cosmos. Everybody else plagiarized.



Queen Nefertiti & Pharaoh Amenhotep

iv. Acknowledgments

UT Austin is different ~ extremely different. Regular Ph.D students have three years from the time they start grad school to take their qualifying exams. Disabled students have two semesters to prepare. Regular grad students have to be put on academic probation and counseled before they are dismissed, but only gently after a few documented semesters of slow progress. Disabled students can be expelled immediately, without recourse; prevented even from continuing as a non degree seeking undergraduate ~ and a disabled student's "slow progress" is judged relative to the aforementioned accelerated schedule, condensing three years of work into two semesters.

It doesn't matter if disabled students have good grades. I had a 3.7 GPA on all the coursework required for a Ph.D, and I passed my written qualifying exams. It doesn't matter if a grad student is a licensed Professional Engineer; if he publishes two engineering textbooks with McGraw-Hill (the largest technical publisher in the world) while he's enrolled in grad school. Here's the University's decision, in the immortal words of tyranny ~ after I complained that I could not get a job because they barred me from contacting anybody at the University involved in my appeal, preventing me from getting references to complete my research at another University - or to get any kind of job, anywhere. Here's a verbatim quote from an email sent to me by Dr. Lee Smith, the UT Vice President for Legal Affairs:

"…letters of recommendation are voluntary honest expressions of the personal opinions of the writer. No faculty member can be compelled to provide a letter of recommendation and no faculty member can be compelled to say positive things that they do not honestly believe about a person's scholarship and compliance and progress and cooperation. The faculty members in our College of Engineering that are familiar with your lack of scholarship, lack of compliance, lack of progress and lack of cooperation with the reasonable requirements of your supervising faculty, have all stated that they do not choose to give you a letter of recommendation. This is their choice. It strikes me that you want letters of recommendation, you will have to find someone outside of The University of Texas that holds you in sufficiently high regard that they can honestly recommend you, because I am not aware of anyone here that fits that description.

I. Gravity

TOPICS: Kepler's Laws
Newton's Laws of Motion
The Two Body Equation
Gravity Field Force

Historical Review.

Aristotle (384-322 BC) was one of the first to believe in a geocentric or earth-centered Solar System. He also believed that bodies of different weights "fell" at different speeds. Other Greek philosophers of this era - most notably Pythagoras - taught a heliocentric or sun-centered Solar System.

The Egyptian scholar Claudius Ptolemaeus (151-127 BC) or Ptolemy published a geocentric model of the Solar System, a complex mechanism of cycles (orbits) and epicycles that was the norm for the next 1500 years. Although the model is usually attributed to Ptolemy alone, most of the work was done by the Egyptians before him. Ptolemy refined some of the tables and added some new work, then published the entire volume as the equivalent of a textbook in astronomy.

In the modern day, everybody disregards Ptolemy's model as almost a joke. However, it was accurate enough for navigating the oceans of the world for almost two thousand years ~ to the farthest extent of latitude and longitude. The model itself may be complicated, intricate, and convoluted. It's accuracy is undeniable, though, and it's hard not to suspect that there may be some kind of validity to it ~ that there is some subtle aspect of gravity that it silhouettes, however inexpertly.

[NOTE: Now and then the reader's attention is drawn to certain anomalies, inconsistencies, and curiosities in current theory. These comments are intended to challenge the reader's intellect. To a limited extent, they lead up to a couple of speculative papers in the Appendix. These papers exhibit a very subtle substructure to the Solar System that has many similarities to Ptolemy's model, while incorporating Relativistic phenomena in a way that standard Newtonian models of orbital motion cannot… leading all together to a dramatic new perspective on gravity.]

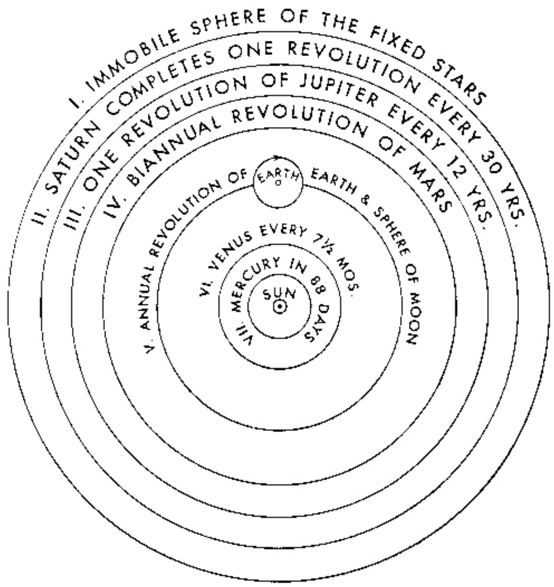


Illustration from Copernicus' Revolutions of Heavenly Spheres

Nicolaus Copernicus (1473-1543) was the first scientist to brave the heliocentric view. He did so tactfully, saying the sun centered model was much simpler than Ptolemy's model and also that it was a more convenient way to calculate the orbits of the planets. Copernicus' model still had small epicycles and slight variations in uniform motion about a central point, but it was one step closer to solving the problem.

The Danish astronomer Tycho Brahe (1545-1601) was court mathematician in Prague, Czechoslovakia. He used the new telescope invention to record accurate positions of the observable planets, but did not offer any new theories on planetary motion.

One of the astronomers who refined the telescope used by Tycho Brahe was the Italian inventor and mathematician Galileo Galilei (1564-1642). He believed in the heliocentric Solar System more fervently than Copernicus, and was put under house arrest by the Church ~ and remained so for the last eight years of his life. Galileo published verbal statements of Newton's first two Laws of Motion, and introduced many new practical ideas in kinematics and dynamics.

The German astronomer Johannes Kepler (1571-1630) succeeded Tycho Brahe as court mathematician in Prague, and inherited all of his predecessor's painstaking observations. Kepler believed the heliocentric model, and using Brahe's data showed the planets were in eliptical orbits around the sun.

Kepler's geometric laws of motion explained the kinematics of planetary motion but it was Isaac Newton (1642-1727) who developed a fundamental, theoretical basis - a law of force to explain the elliptical motion of the planets.

Epicycles

Copernicus' heliocentric model of the Solar System had circular orbits for the planets, with small epicycles for some of the more elliptical orbits. Just as Newton's term for momentum was motion, what would be the modern term for epicycles makes that arcane term quite brilliant.

Fourier Series are what we call epicycles these days. A small eccentricity ellipse can be closely approximated by adding a small amplitude sine wave to the unit circle, The sine wave has a wavelength equal to the circumference of the unit circle, 2pi. An even more exact match is possible by adding another (cosine) wave with half the wavelength - equal to pi - and so forth. The more eccentric the shape, the more terms necessary but a consummate accuracy is eventually achieved. This is the concept of Fourier Series. The same process can be done to approximate small inclinations of an orbit, adding the displacement perpendicular to the orbital plane.

A common way to approximate planetary orbits in computer algorithms is to adjust the inclination and eccentricity using sine/cosine functions as just described. The resulting almanac of planetary positions is actually quite satisfactory for amateur astronomers, in finding planets in the night sky.

Kepler's Laws

Kepler's first two laws were published in 1609 and the third law in 1619. An easy way to remember them is to number them as indicated:

0. The orbit of each planet is an ellipse with the Sun at one focus

1. A line adjoining the planet to the Sun sweeps out equal areas in equal times

2,3. The ratio of the squares of the periods of two planets is equal to the ratio of the cubes of the semi major axis of their orbits.

Kepler derived these laws geometrically. They can also be derived analytically from Newton's Laws and calculus.

Newton's Laws

The principals of this book can be traced directly to Newton's three Laws of Motion. The first two laws were known to both Galileo and Hyghens, but were first announced by Newton all together in Principia.

I. Every body continues in a state of rest, or of uniform motion in a straight line, unless it is compelled to change that state by a force impressed upon it

II. The rate of change of motion is proportional to the force impressed, and takes place in the direction of the straight line in which the force acts

III. To every action there is an equal and opposite reaction; or, the mutual actions of two bodies are always equal and oppositely directed

Newton's Law of Gravitation was also discussed in Principia. The gravitational force acting between two bodies is proportional to the product of the masses and is inversely proportional to the square of the distance between them. In vector form the equation is

$$\vec{F} = G\frac{m_1 m_2}{r^2}\hat{r}$$

Where G is the universal constant of gravitation
$\hat{r}$ is a vector of length one in the direction of $\vec{r}$

A few remarks provide valuable insight into Newton's laws, as they were posed and developed in the scientific language of the time.

1. The Laws of Motion were developed using geometry and not calculus

2. Newton's term for momentum is motion. Force, in the second law, is the rate of change of momentum

$$\vec{F} = \frac{d}{dt}\left(m\vec{v}\right)$$

3. The second law implies simultaneity - the change of motion resulting from several forces is the same as if each force acted separately.

4. The parallelogram of forces was also derived by Newton from the second law: the resultant of two forces is represented by the diagonal of a parallelogram, the two forces being adjacent sides. This directed line is a vector.

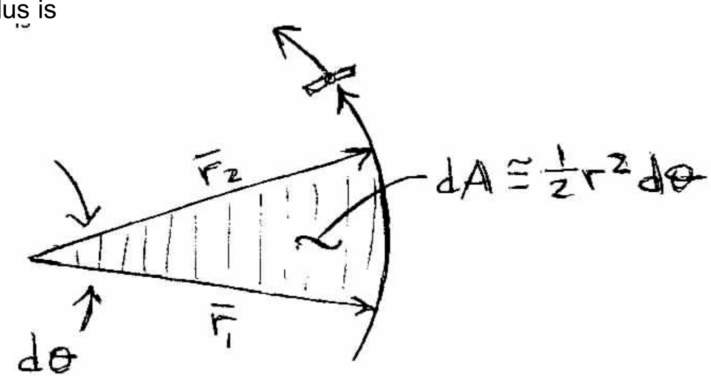
5. It is interesting to note that the electrostatic force at the quantum level is also an inverse squared force:

$$\vec{F} = k\frac{q_1 q_2}{r^2}\hat{r}$$

This law can be derived geometrically exactly as Newton derived the Laws of Motion.

Law of Areas

Newton had invented calculus before he published <u>Principia</u>. One of the proofs done geometrically was to prove Kepler's Law of Equal Areas. It's not difficult to show how using the calculus to derive this law might have led to controversy. The geometry used to develop the formula using calculus is



Strictly speaking, the length of the radius vectors on either side of the area are not equal if the body is in an elliptical orbit. A more precise definition is to substitute $r_1 r_2$ for r^2 in the equation for area.

Newton's geometric derivation in Principia did not make this limiting assumption, and therefore avoided any possible controversy.

Theory of Orbits

In the following hundred years after Newton, virtually every famous mathematician made contributions to the ancient science of celestial mechanics, most notably in the development of the Three Body Problem (3BP) theory. The stability points for the circular coplanar 3BP are named after Lagrange (1736-1813), who published a paper deriving them in 1772. Euler (1707-1783) arrived at the same result as Lagrange later the same year. Jacobi (1804-1851) found the exact integral for the energy in a restricted 3BP in 1836, and Poincare (1854-1912) proved in 1899 that no other exact integrals were possible for the 3BP. The Lagrange equilibrium points were proved when in 1906 small groups of asteroids were observed in orbit around the L4 and L5 points of the Sun-Jupiter system ~ the Trojan and Apollo asteroid groups.

Whole branches of modern mathematics arose from attempts to solve the intractable 3BP and other challenges of celestial mechanics in the 20^{th} century. Here Fourier Series found one of their first practical applications. Perturbation techniques were developed by workers in celestial mechanics, then used to solve other dynamical problems. And so forth.

Most theoretical work in astronomy is now done using computer models and statistics. Results have been impressive, but statistical solutions often lack the insight and simplicity of the analytical solution. The original fundamental theory is still in common use, however, as the basis for statistical models and theories.

The Two Body Equation

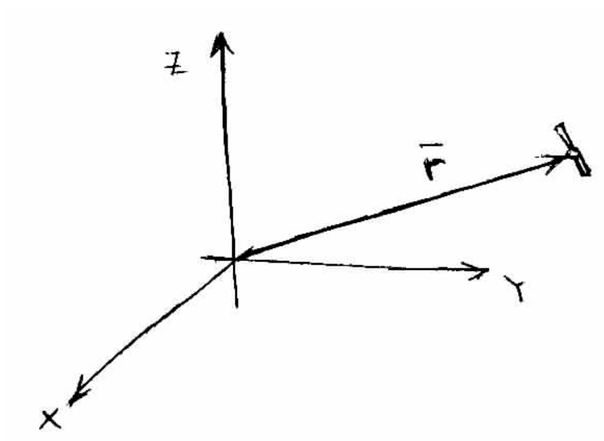
Newton's Universal Law of Gravitation refers to the force between two masses ~ the so called Two Body Problem. A complete understanding of the 2BP is vital to the mathematical description of orbital motion and is the foundation for all statistical computer models as well.

If you combine the notion for force/momentum in Newton's 2nd law with the principle of simultaneity implied in the 3rd law, you get the force equation

$$\sum \vec{F} = \frac{d}{dt}\left(m\vec{v}\right) = m\vec{a}$$

where $\vec{v} = \ddot{\vec{r}}$ and $\vec{a} = \ddot{\vec{r}}$

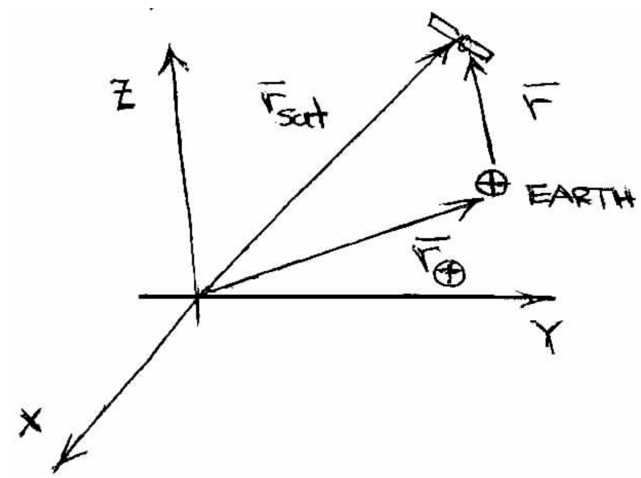
Consider the 2BP of an Earth satellite, with m_1 the Earth and m_2 the satellite. Placing Earth at the origin of a Cartesian coordinate system, with a directed line segment or vector toward the satellite, it is possible to use the principles developed so far to derive an equation for the motion of the satellite, or second body.



Assuming an inertial coordinate system and point masses, the force of Earth's gravity acting on the satellite can be expressed

$$\vec{F}_g = -G\frac{m_{\oplus} m_{sat}}{r^2}\hat{r} \text{ where } \hat{r} = \frac{\vec{r}}{r}$$

To find the satellite's differential equations of motion, consider the new coordinate system.



In vector form, $\vec{r} = \vec{r}_{sat} - \vec{r}_{\oplus}$ and the forces between Earth and the satellite are

$$\vec{F}_{sat} = m_{sat}\vec{r}_{sat} = -G\frac{m_{\oplus}m_{sat}}{r^2}\hat{r}$$

$$\vec{F}_{\oplus} = m_{\oplus}\vec{r}_{\oplus} = +G\frac{m_{\oplus}m_{sat}}{r^2}\hat{r}$$

Taking a second derivative of the position vector equation,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_{sat} - \ddot{\vec{r}}_{\oplus}$$

Solving for $\ddot{\vec{r}}_{\oplus}$ and $\ddot{\vec{r}}_{sat}$ from the force equations and substituting,

$$\ddot{\vec{r}} = -G\frac{\left(m_{\oplus} + m_{sat}\right)}{r^2}\hat{r}$$

Assuming the satellite mass is negligible compared to the mass of the Earth,

$$\ddot{\vec{r}} = -\frac{Gm_{\oplus}}{r^2}\hat{r} \text{ for } m_{sat} \ll m_{\oplus}$$

Where $Gm_{\oplus} = \mu$, the gravitational constant for Earth, and

$$\boxed{\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}}$$

This is called the Two Body Equation. Note carefully the assumptions used to derive this important equation:

0. The satellite mass is many orders of magnitude less than the mass of Earth and is therefore neglected

1. No other forces (e.g. drag) act on the satellite other than the Earth's gravitational attraction

2. The two bodies are spherically symmetric and can be represented as point masses

3. Motion takes place in an inertial coordinate system

(the numbering system is there to help remember, by analogy)

The model of the atom used by Niels Bohr came from the Two Body Equation. It's one proton in the nucleus and one electron. Bohr's Ph.D dissertation in 1916, posing this as a working model of the atom, was the beginning of subatomic physics and the nuclear age.

Gravity Field Force

Another way to derive the Two Body Equation is to use the gradient of the Potential, U. That is,

$$U_{po\text{int}\,mass} = \frac{\mu}{r}$$

This is the equivalent, in dynamics theory, of potential energy. The radius vector in an inertial system is

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

From which the magnitude of the radius vector is

$$r = \sqrt{x^2 + y^2 + z^2}$$

Assuming U is a vector, it is possible to take it's gradient,

$$\nabla U = \frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}$$

Substituting into the point mass equation for U and taking the partial derivatives,

$$\nabla U_{po\text{int}\,mass} = -\frac{1}{2}\frac{\mu}{\left(x^2 + y^2 + z^2\right)^{3\backslash 2}}\left[2x\hat{i} + 2y\hat{j} + 2z\hat{k}\right]$$

Substituting for the functions of $\vec{r}$ and r, you get

$$\boxed{\nabla U_{po\text{int}\,mass} = -\frac{\mu}{r}\vec{r}}$$

Thus $\nabla U = \ddot{\vec{r}}$ and you have the Two Body Equation.

This is the model used to study the gravitational field of the Earth. U for a non-uniform body is the sum of many contributions of finite mass

$$\iiint \frac{G}{R}\, dm[\;\;]$$

Where the quantity in brackets are Lagrange polynomials, G is the gravitational constant of the Earth, and R is Earth's nominal radius.

A detailed model of Earth's gravity field, solving all the polynomial terms, takes an enormous amount of computer power ~ solving a 1000x1000 matrix or larger. In these models G and R are also input as unknowns, and the numbers quoted in standard references are the values calculated from these gravity models.

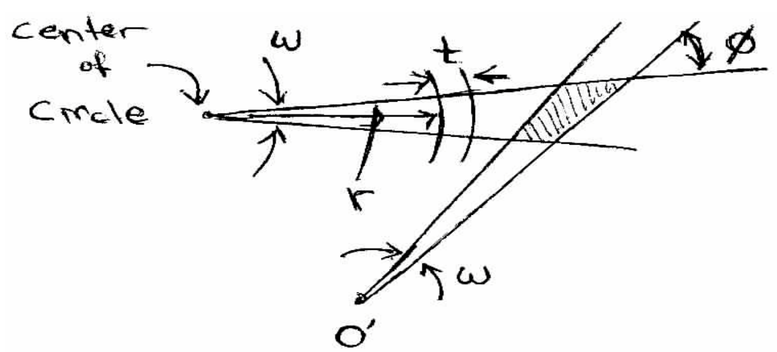
$$G = 3.986004 \text{ E5 km}^3/\text{sec}^2$$
$$R = 6378.1363 \text{ km}$$

Notice that these values are determined by statistical means. As the gravity model becomes more elaborate, the values change - perhaps only in say the sixteenth significant digit; but they are in fact arbitrary.

This is not a problem for simple Two Body calculations, but when it comes to Relativistic calculations we will see that as slight an error as one digit in the twentieth significant digit can adjust the inertial coordinate system enough to negate any observed Relativistic phenomena. Not to mention round off and truncation errors, when performing so many billions of calculations. It is not hard to make a case that no Earth gravity models are accurate enough to confirm relativistic phenomena in Earth satellites. More on this later.

Gravity in a Shell

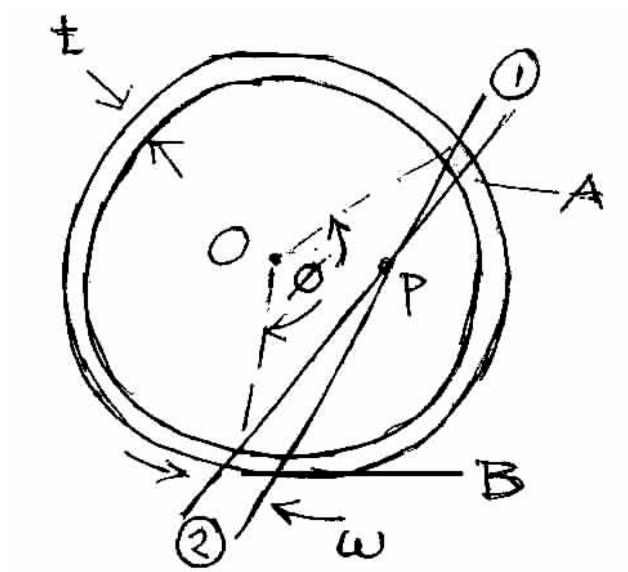
In the modern day, people are skeptical about Newton using geometry to derive the Laws of Motion. If anything thing, it should be the other way around ~ calculus is less accurate than geometry. Consider the following sketch of Newton's proof for "The Attraction of a Thin Homogeneous Spherical Shell upon a Particle in its Interior" to show the accuracy of geometry. Consider the volume within a small cone, between two spherical surfaces whose centers are at the vertex of the cone



The volume is defined by

$$Vol = \frac{\omega r^2 t}{\cos\phi}$$

Now consider a thin spherical shell and find the force on a point in the interior not at the center.



The volumes at (1) and (2) are

$$V_1 = \frac{\omega t \overline{AP}^2}{\cos\phi} \text{ and } V_2 = \frac{\omega t \overline{BP}^2}{\cos\phi}$$

multiply both by a density σ to get the masses of the elements, and the attractions by the Universal Law of Gravity are therefore

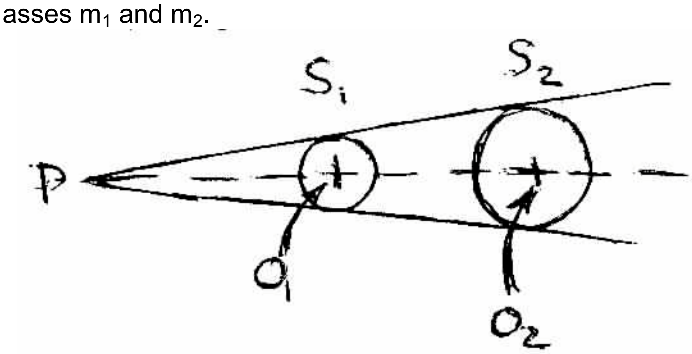
$$f_1 = \mu\sigma \frac{\omega t \overline{AP}^2}{\cos\phi \overline{AP}^2} \text{ and } f_2 = \mu\sigma \frac{\omega t \overline{BP}^2}{\cos\phi \overline{BP}^2}$$

The forces are equal. Likewise for every other small solid angle at P. Consequently, the sum of all forces at P is zero. Likewise for any number of thin spherical shells, and so for shells of finite thickness.

This same kind of analysis works for ellipsoidal shells - the attraction of a thin elliptical homeoid on an interior particle is zero. This is hugely complicated to do using calculus and triple volume integrals, but not much more difficult than for the above sphere using geometry.

The Point Mass Assumption

Consider two shells internally tangent to a cone and an exterior point with masses m_1 and m_2.



Where a_1 = PO_1 and a_2 = PO_2

The two shells attract a particle at P equally because any solid angle which includes part of one sphere includes a similar part of the other sphere; the included masses are as the squares of their distances, and their attractions are inversely as the square of their distances.

Now move S_1 to be centered on O_2 and let A_1 be the attraction of S_1 on the point; likewise for A_2.

$$\frac{A_1}{A_2} = \frac{a_1^2}{a_2^2} = \frac{M_1}{M_2}$$

So the two spheres attract a particle at the same distance with a force directly proportional to their masses. Therefore, a sphere which is homogenenous in concentric layers acts on an exterior particle as if all the mass is at its center.

Observe, the point mass assumption is not just for a homogeneous sphere ~ but if any given layer is homogeneous, i.e. all composite spheres need not be of the same density; only individual shells of infinitesimal thickness need be of constant density.

Given the generally layered composition of Earth, this proof extends the point mass assumption to a much large class of bodies, and it makes the Two Body model much less simplistic than it might seem at first. The problem is that the Lagrange polynomial model does not take advantage of this natural symmetry of the Earth and so, including all it's other assumptions and inconsistencies, it's accuracy is limited. Considering all the uncertainties of the model itself and the computational solution, it's not difficult to disqualify any claim to relativistic phenomena in Earth satellites.

The Almagest

The complicated heliocentric model of planetary motions created by Ptolemy was made to satisfy quite strict criteria. The defining principles of Ptolemy's model of the Solar System are specified by the two rules:

1. Every planet exhibits uniform motion on a circle

2. With or without the uniform motion of the center of that circle

This is exactly as stated, in rotating coordinates, the Fourier Series Theory ~ that any curve can be the created by the sum of sines and cosines.

The motions of the planets are extremely complicated even in a sun centered model, with the sun fixed in space. The idea of explaining the motions of the planets with respect to the moving Earth is a absurd in the extreme. It's so extremely unlikely that such a simple mechanism as Ptolemy's could work at all (much less with quite good accuracy), that there must result an undeniable conclusion that there truly is some symmetry with respect to Earth. Moreover, this model also gives the risings and settings of the Sun and planets ~ well, the planets known at the time, Mercury, Venus, Mars, Jupiter, and Saturn.

II. Unknown Variables

TOPICS: Angular momentum
Energy integral
Center of mass
Barycenter

Ten Unknowns

This section develops a mathematical definition for the ten known variables of the Two Body Problem.

3	$\vec{h} = \vec{r} \times \vec{v}$	angular momentum integral
1	$E = \dfrac{v^2}{2} - \dfrac{\mu}{r}$	energy integral
6	$\vec{r}_{cm} = \vec{A} + \vec{B}t$	position vector

———————

10 Unknowns

Both "integrals" as they are called in celestial mechanics, are from the Two Body Equation.

Angular Momentum

An expression for angular momentum $\vec{h}$ comes from taking the cross product of the velocity vector and the Two Body Equation:

$$\vec{r} \times \left[\ddot{\vec{r}} + \frac{\mu}{r^3} \hat{r} \right]$$

$$\vec{r} \times \ddot{\vec{r}} + \vec{r} \times \frac{\mu}{r^2} \hat{r} = 0$$

The second term is zero because $\vec{r} \times \vec{r} = 0$, leaving

$$\vec{r} \times \ddot{\vec{r}} = 0$$

Now observe that

$$\frac{d}{dt}\left(\vec{r} \times \dot{\vec{r}}\right) = \dot{\vec{r}} \times \dot{\vec{r}} + \vec{r} \times \ddot{\vec{r}} = 0$$

Where the first term $\dot{\vec{r}} \times \dot{\vec{r}}$ is equal to zero, leaving

$$\frac{d}{dt}\left(\vec{r} \times \dot{\vec{r}}\right) = 0$$

Integrating,

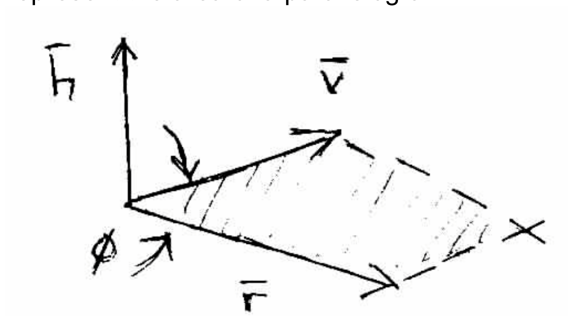
$$\vec{r} \times \dot{\vec{r}} = \vec{h}$$

Where $\vec{h}$ is the constant of integration. Since $\vec{r}$ and $\dot{\vec{r}}$ both lie in the orbital plane, $\vec{h}$ is perpendicular to this plane. The convention in dynamics is to call this vector the angular momentum vector.

$$\boxed{\vec{h} = \vec{r} \times \vec{v}}$$

Flight Path Angle

The magnitude of the cross product of two vectors is known from analytic geometry to represent the area of a parallelogram.



The included angle is the Flight Path Angle ϕ. It is interesting to note that the direction and magnitude of $\vec{h}$ is constant no matter where you are in an elliptical orbit, but the area that is the magnitude of $\vec{h}$ changes.

The radius and velocity vectors are orthogonal at all points on a circular orbit, so the area is a rectangle and this shape rotates around the z axis at a constant rate. In an elliptical orbit, the area is rectangular only at the nearest and farthest points when the radius and velocity are perpendicular. The area rotates with the radius vector, changing the most at periapse (the closest point to the central body) and slowest at the most distant point, apoapse.

As with the Law of Equal Areas analysis, this example suggests $\vec{h}$ may have constant direction and magnitude, but some subtle aspect of it varies. This idea will be developed as the narrative advances through the chapters. Ultimately the objective is to show how Relativistic phenomena are evidenced even in fundamental Two Body theory. That is, Relativity modifies Newton's Law of Gravity thus:

$$\vec{F} = -G\frac{m_1 m_2}{r^2}\hat{r} + \dot{\vec{r}}\,\frac{\vec{F}\bullet\vec{r}}{c^2}$$

The second term is divided by the speed of light squared, so it's clearly negligible in normal space time. However, mathematically it should still be evident in some way, shape, or form in the Two Body Problem.

The Energy Integral

The second integral of motion for two body motion is to take the dot product of the velocity into the Two Body Equation

$$\dot{\vec{r}}\bullet\left[\ddot{\vec{r}} = -\frac{\mu}{r^2}\hat{r}\right]$$

$$\dot{\vec{r}}\bullet\ddot{\vec{r}} = -\frac{\mu}{r^3}\vec{r}\bullet\dot{\vec{r}}$$

Observe that both terms in the second equation have an $\dot{\vec{r}}$ term and since $\ddot{\vec{r}}$ is the gravitational acceleration between the two bodies, it is in the same direction as $\vec{r}$. SO the vectors can all be changed to scalars after the dot products are removed (i.e. the $\cos\theta$'s cancel out)

$$\dot{r}\ddot{r} = -\frac{\mu}{r^3}\dot{r}r = -\frac{\mu}{r^2}\dot{r}$$

Now observe the derivatives,

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = v\dot{v} = \dot{r}\ddot{r}$$

$$\frac{d}{dt}\frac{\mu}{r} = \frac{\mu}{r^2}\dot{r}$$

Substituting,

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) + \frac{d}{dt}\left(-\frac{\mu}{r}\right) = 0$$

Integrating this function, you get a constant of integration, which by convention is called E for energy because the two terms have units for kinetic and potential energy.

$$\boxed{E = \frac{v^2}{2} - \frac{\mu}{r}}$$

The energy E is constant at all points on a given Two Body orbit, so this provides one more variable because E is a scalar.

The last six variables for the Two Body Problem come from the assumption that the analysis takes place in an inertial reference frame. (Actually, this is a requirement by Newton's Laws of Motion.) This means that the center of mass of the two bodies must either be at rest or moving at a constant velocity.

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$

Differentiate twice to get accelerations

$$\ddot{\vec{r}}_{cm} = \frac{m_1\ddot{\vec{r}}_1 + m_2\ddot{\vec{r}}_2}{m_1 + m_2} = 0$$

This must equal zero because $m_1\ddot{\vec{r}}_1$ and $m_2\ddot{\vec{r}}_2$ are equal and opposite. Consequently,

$$\ddot{\vec{r}}_{cm} = 0$$

$$\dot{\vec{r}}_{cm} = \vec{B}$$

$$\vec{r}_{cm} = \vec{A} + \vec{B}t$$

Showing the center of mass moves in a straight line at a constant velocity. The vectors $\vec{A}$ and $\vec{B}$ comprise six unknowns, and now the ten variables of the Two Body Problem are found ~ these six plus one for the Energy plus three for the angular momentum vector.

Center of Mass

It is possible to derive the equations of motion for two masses in terms of vectors from the center of mass of the system of two masses.

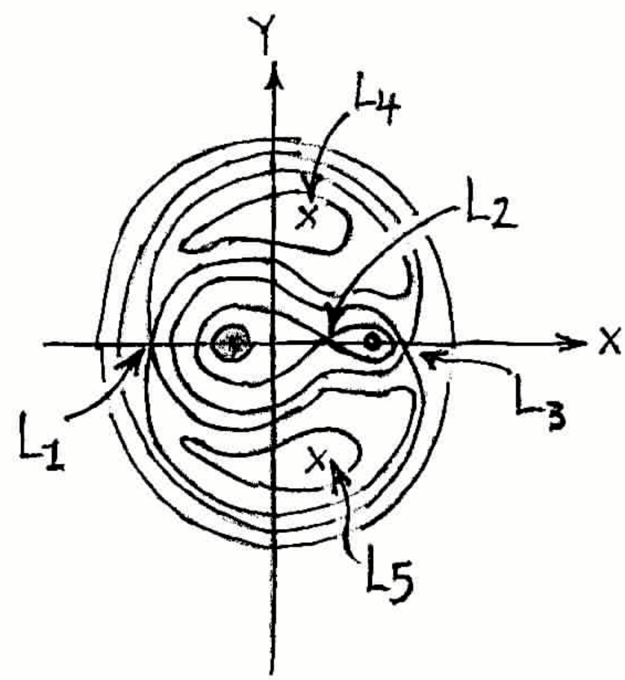
$$\ddot{\vec{\rho}}_1 = -G\frac{m_2^3}{\left(m_1 + m_2\right)^3}\frac{\vec{\rho}_1}{\rho_1^{\,3}}$$

$$\ddot{\vec{\rho}}_2 = -G\frac{m_1^3}{\left(m_1 + m_2\right)^3}\frac{\vec{\rho}_2}{\rho_2^{\,3}}$$

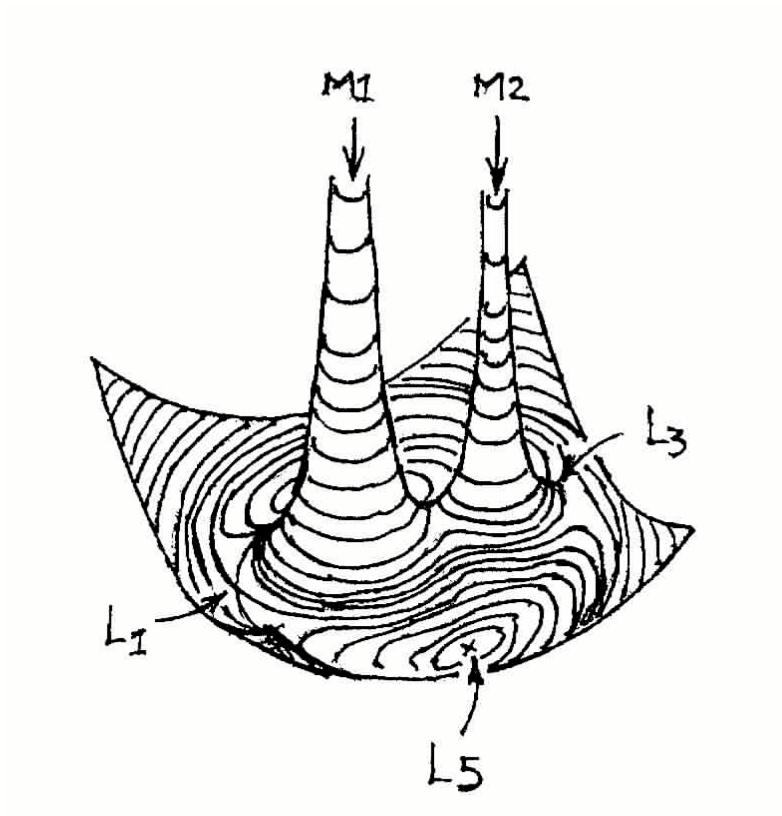
Where $\vec{\rho}_1$ is a vector from the center of mass to mass #1 and $\vec{\rho}_2$ is from the center of mass to mass #2.

This derivation is included because the result is similar to the Three Body Problem or 3BP (two large primaries and a much smaller third mass), with two equal and opposite forces acting at the center of mass. In the above analysis the center of mass is just a point; in the 3BP it can be occupied by a very small mass. The 3BP is usually analyzed in rotating coordinates (the line between the two large masses rotating about the center of mass ~ the origin of the xy coordinate system), which introduces two new forces ~ corriolis acceleration and centrifugal acceleration.

This kind of rotating system is the norm in nature, and not the exception ~ at least when it comes to celestial mechanics. The existence of these two new forces introduces a massive complexity to the problem, such that the third small body can occupy any number of stable positions.



In the above illustration, you can see that a whole family of elliptical orbits exists around each of the Lagrange points, both in and out of the xy plane. As the energy of the third body increases, the available orbits diminish until they are limited to motion around L4 and L5 ~ energy is the vertical axis; so low energy orbits can only be around either of the two masses; higher energy orbits can be around both; then higher still are around just the L4 and L5 Lagrange Points; finally, the particle escapes the two mass system all together.



In a manner of speaking, these are all the lines of force that exist between two bodies. The stable orbits along continuous lines are all "safe" orbits for a third small body, where the three forces balance ~ gravity, centrifugal, and corriolis forces. The latter two forces are "hidden" in the Two Body analysis, but in actual circumstances of rotating motion they are very much in evidence. Hopefully these figures show how complex the interplay of forces is between two masses, and makes it seem less far fetched that Relativistic phenomena might possible evidence themselves in Two Body theory.

Sphere of Influence

The figure 8 shaped orbit around the two primaries in the first illustration, intersecting at L2 is called a "free return orbit." The early Apollo missions were on this orbit in the Earth-moon system because a spacecraft that lost power, say, on the far side of the moon would drift right back towards Earth on the figure 8, without any further thrusts or flight path corrections. That actually happened with Apollo 13.

It is interesting to note that tens of thousands of computer simulations of the Earth to Moon trajectory were done in the early years of the space race, but not one of them found the very fast and efficient - and safe - free return orbit indicated. It was only found by a mathematical study of the 3BP, in which that orbit is quite obvious as you can see.

The point origin of the xy coordinate system is the center of mass between the two primaries. If you create a sphere of this radius around the larger body (e.g. Earth), this is called the Sphere of Influence. The distance is proportional to the mass ratio by the following relationship

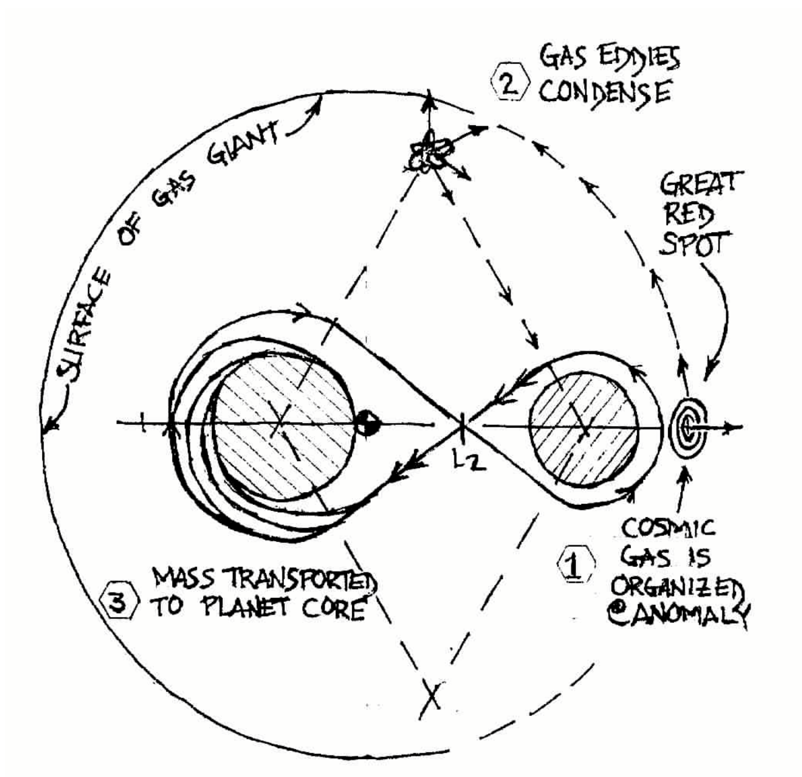
$$r_{soi} = \rho_{12}\left(\frac{m_1}{m_2}\right)^{2/5}$$

where ρ_{12} is the distance between the two large masses.

Later, in studying the 3BP, it will be shown that L1, L2 and L3 are unstable equilibrium points. A small mass can remain at any of those points but quickly moves away if any external forces act on it. However, if the small mass is in a small elliptical orbit perpendicular to the orbit of the primaries about either of those three points, it will remain in that orbit even in the presence of external forces or perturbations. These are called "halo orbits," and are used for many practical purposes. Again, computer models never found those halo orbits; only mathematical analysis brought them to light.

This is intended to illustrate the limitations of computer models and statistical analyses. They give no intuitive understanding of a problem, and therefore are limited in revealing subtle nuances to the interplay of forces. On the other hand, the Lagrange Points and the stable orbits associated with them are quite clearly shown by the mathematics ~ and these are only a few of hundreds of even more delicate features revealed by mathematical analysis.

To put this all into perspective, consider the illustration below. It's my theory about the Red Spot on the gas giant Jupiter (there's another dark spot on Neptune, also a gas giant). The diagram shows two large masses within the core of the planet, and how the surface spot could be just a halo orbit relative to the two mass anomalies. Actually, for a body that is all gas, there is no other way that a permanent surface feature could exist than as orbits around a stable point, as indicated.

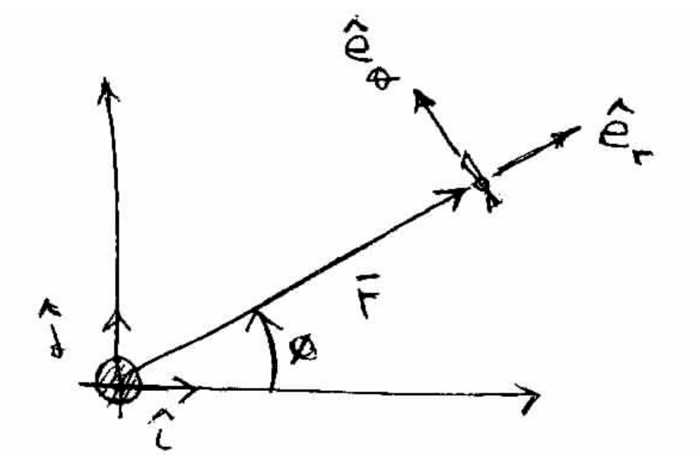


The Energy Integral

It is opportune to derive the Energy Integral again, using a more rigorous method. This kind of strict derivation will be used later in the narrative and it's easier to follow later if you get accustomed to it now, on a simple problem. As before, the derivation begins by taking the dot product of the velocity vector into the Two Body Equation.

$$\dot{\vec{r}} \bullet \ddot{\vec{r}} + \dot{\vec{r}} \bullet \frac{\mu}{r^3}\vec{r} = 0$$

This time, consider $\vec{r}$ as a vector in the Cartesian coordinate system represented by the unit vectors $\hat{i}$ and $\hat{j}$ where $\vec{r}$ has radial and tangential local coordinates of $\hat{e}_r$ and $\hat{e}_\theta$.



$$\vec{r} = r\cos\hat{i} + r\sin\hat{j} = r\hat{e}_r$$

$$\dot{\vec{r}} = \left(\dot{r}\cos - r\sin\right)\hat{i} + \left(\dot{r}\sin + r\dot{\theta}\cos\right)\hat{j}$$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta \qquad \text{where} \qquad \hat{e}_\theta = -\sin\hat{i} + \cos\hat{j}$$

Now, taking the time derivative of $\hat{e}_r$

$$\dot{\hat{e}}_r = \left(-\sin\hat{i} + \cos\hat{j}\right)\dot{\theta}$$

$$= \hat{e}_\theta \dot{\theta}$$

Calculating, $\vec{r} \bullet \dot{\vec{r}}$

$$\vec{r} \bullet \dot{\vec{r}} = r\dot{r}$$

And substituting, you get

$$\dot{\vec{r}} \bullet \ddot{\vec{r}} + \mu \frac{\dot{r}}{r^2} = 0$$

As before, notice the derivatives:

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) = v\dot{v} = \dot{r}\ddot{r}$$

$$\frac{d}{dt}\frac{\mu}{r} = \frac{\mu}{r^2}\dot{r}$$

Substituting,

$$\frac{d}{dt}\left(\frac{v^2}{2}\right) + \frac{d}{dt}\left(-\frac{\mu}{r}\right) = 0$$

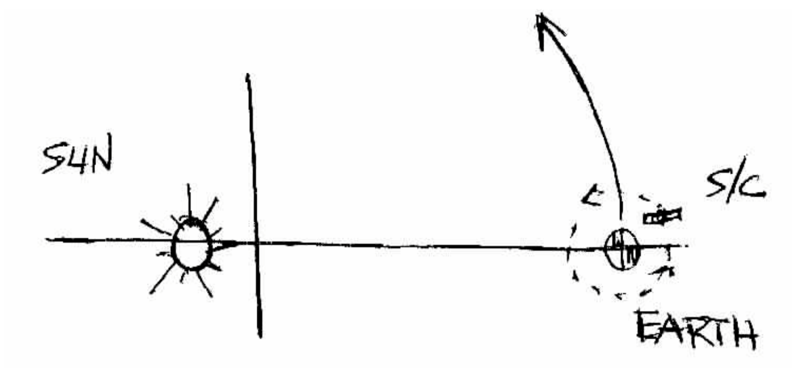
Integrating and setting equal to a constant of integration E, you get

$$E = \frac{v^2}{2} - \frac{\mu}{r}$$

Barycenter

There is another energy integral for the circular coplanar 3BP - i.e. two bodies in circular orbits in the same plane, with the third body being much smaller than the primaries, like a spacecraft in the Earth-Moon system. In such a model, the energy of the third small body is constant along its trajectory (i.e. lines in the previous illustrations). However, it is constant only when calculated in an inertial reference frame.

Three Body Problems consider the motion of spacecraft or other small body, relative to two massive primary bodies like Earth and the Sun. A spacecraft on an interplanetary mission from Earth to Mars does not have constant energy in the 3BP relative to the Sun, if you place the origin of the coordinate system at the center of the Sun. This is not an inertial reference frame because the center of mass of the Earth-Sun system is not at the center of the Sun.



A 3BP typically is in rotating coordinates, with a line between the Sun and Earth, say, rotating at a constant velocity ~ just like Earth orbits the Sun in reality. In this case, the center of mass or barycenter is as indicated in the figure ~ wherein the Sun itself is in a small orbit around the center of mass and consequently does not experience constant velocity in a straight line ~ the stipulation for an inertial reference frame.

The importance of choosing an inertial reference frame is quite obvious when modeling interplanetary scenarios. If the energy integral - called the Jacobi integral in a 3BP - is calculated versus the Sun at the origin, the energy varies greatly; if the Jacobi integral is calculated versus barycenter, it's constant just like the theory says.

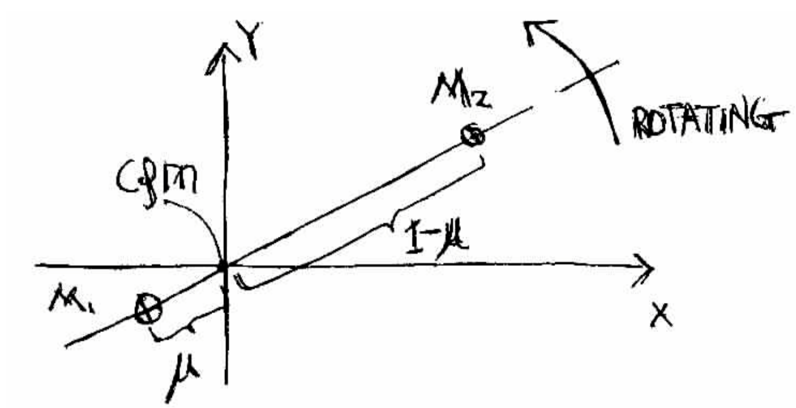
Most applications of the 2BP do not worry about an exact center of mass or inertial reference - not until the more advanced topics or applied orbital mechanics are considered, where the sun and planets are perturbing forces; there's drag and other forces.

A spacecraft orbiting Earth, the Moon, or Mars is accurately modeled using 2BP theory. However, the shuttle rendezvousing with the International Space Station or a comet is not a good 2BP because the relative mass of the shuttle is not negligible.

A good habit is to set up a coordinate system for a problem involving two large primaries the way it's done for the standard 3BP.

Let $m_2 < m_1$ and $\mu = \dfrac{m_2}{m_1 + m_2}$

Then if $m_2 = \mu$ and $m_1 = 1 - \mu$, the distance between the primary bodies is normalized to 1 and the problem look like this



With the axis between m_1 and m_2 rotating as indicated, their relative position is as if m_2 were rotating m_1 in a circular orbit. However, if motion of a third body is to be studied in an inertial reference frame, about the center of mass at the origin of the xy coordinate system, then in reality both masses are in circular orbits about the origin.

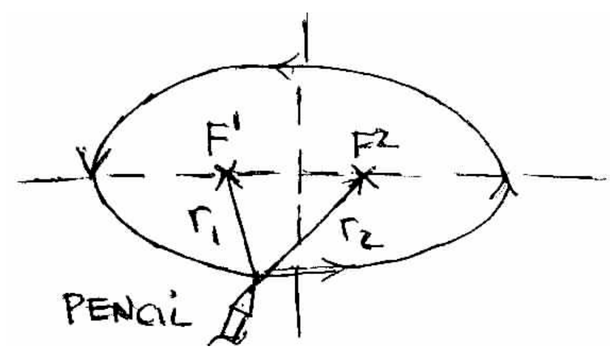
III. Geometry of Conic Sections

TOPICS: The Ellipse
Flight Path Angle
Vis-Viva Equation
The Hyperbola
Velocity Hodograph

The Ellipse

Now to consider the geometry of an ellipse. An ellipse is easy to draw. Attach a fixed length of string to two points, the foci. Stretch the string tight with the point of a pencil and draw a closed figure. Mathematically what this means is that

$$r_1 + r_2 = 2a = constant$$

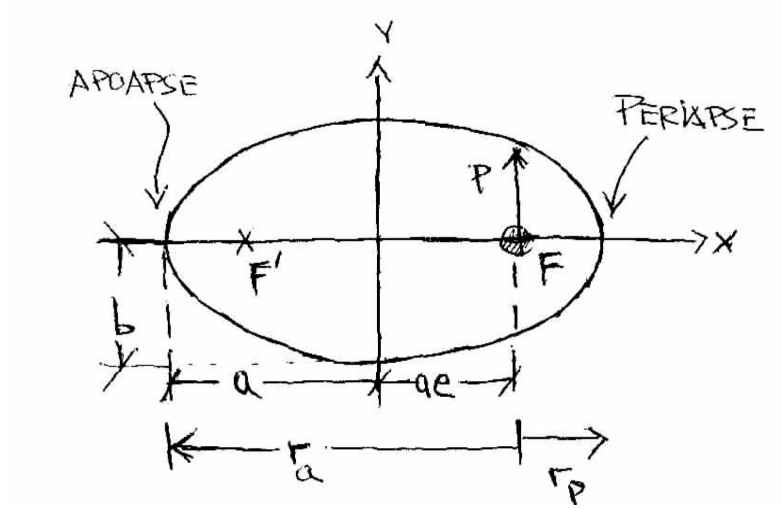


From analytic geometry the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Where the two foci are at (±c,0) and the vertices are at (±a,0) in the (x,y) coordinate system of Figure 1, which has the origin midway between the foci.

Kepler's First Law states that the orbit of each planet is an ellipse with the sun at one focus. Isolating on the sun and one planet – the Two Body Problem – consider the geometry of the ellipse.



The following equations define the dimensions of an ellipse:

$$a^2 = b^2 + c^2 \qquad b = \sqrt{1-e^2}$$

$$e = \frac{c}{a} = \frac{r_a - r_p}{r_a + r_p} \qquad p = \frac{b^2}{a} = a\left(1-e^2\right)$$

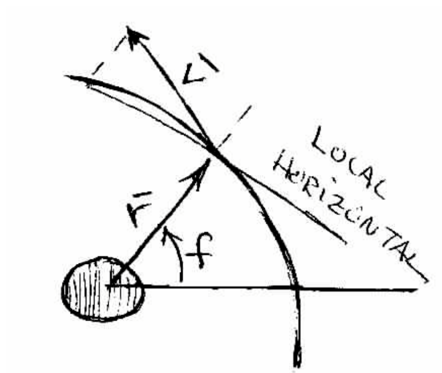
$$r_p = a(1-e) \qquad r_a = a(1+e)$$

where: a = semi-major axis b = semi-minor axis
e = eccentricity p = semi-parameter
r_p = radius at periapse r_a = radius at apoapse

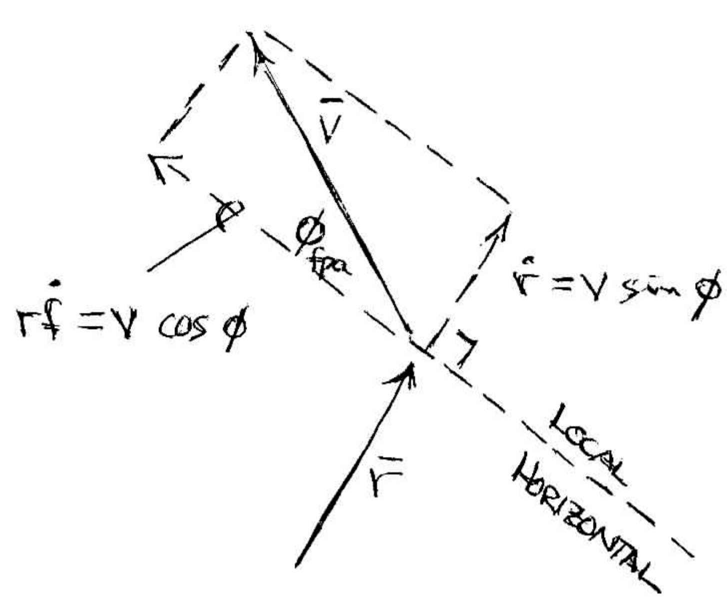
A more convenient reference system in the study of orbits places the origin at the primary focus (c,0) where the central body is situated. The angle of the spacecraft (s/c) on the ellipse is measured with respect to the positive x-axis which goes through the periapse. This angle is called the true anomaly, f.

Vis-Viva Equation

Consider an Earth centered coordinate system,



The angle ϕ_{fpa} is between the local horizontal ($\perp$ to $\vec{r}$) and the velocity vector, where $\phi + \theta = 90°$



Here the velocity vector is broken down into radial and transverse components. Consider the transverse component,

$$v\cos\phi = r\dot{f} = \frac{h}{r}$$

Where $h = r^2\dot{f}$ Observe that ϕ_{fpa} is zero at peripase and apoapse, giving the useful functions

$$h = r_a v_a$$

$$h = r_p v_p$$

These equations help to derive a useful variation of the energy integral. Using the periapse location,

$$E = \frac{v^2}{2} - \frac{\mu}{r} = \frac{h^2}{2r_p^2} - \frac{\mu}{r_p}$$

Where $r_p = a(1-e)$ and in the next section we will derive $h = \sqrt{\mu p}$
Therefore, $h = \sqrt{\mu a\left(1-e^2\right)}$ Substituting, you get

$$\boxed{E = -\frac{\mu}{2a}}$$

Alternately, you can rearrange the equation to obtain

$$v_p^2 = \frac{\mu a\left(1-e^2\right)}{a^2\left(1-e\right)^2} = \frac{\mu\left(1+e\right)}{a\left(1-e\right)}$$

Substituting this into the energy integral

$$E = \tfrac{1}{2}\left[\frac{\mu(1+e)}{a(1-e)}\right] - \frac{\mu}{a(1-e)}$$

$$= -\tfrac{1}{2}\mu\frac{(1-e)}{a(1+e)}$$

Also $E = -\dfrac{\mu}{2a}$ Solving you get a variation on the energy integral called the Vis-Viva equation

$$\boxed{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

From which,

$$v_p = \sqrt{\frac{\mu(1+e)}{a(1-e)}} \text{ and } v_a = \sqrt{\frac{\mu(1-e)}{a(1+e)}}$$

For a circular orbit the eccentricity is zero and $a = r$ Consequently, the circular velocity is defined as

$$\boxed{v_{circ} = \sqrt{\frac{\mu}{r}}}$$

A spacecraft that reaches escape velocity will be on a parabolic escape trajectory. The energy E = 0 for parabolas, so

$$E = \frac{v_{esc}^2}{2} - \frac{\mu}{r} = 0$$

Solving for v_{esc}

$$\boxed{v_{esc} = \sqrt{\frac{2\mu}{r}}}$$

Another useful function comes from $E = -\frac{\mu}{2a}$ Or, $-\frac{1}{a} = \frac{2E}{\mu}$ Recall that $p = \frac{h^2}{\mu} = a\left(1 - e^2\right)$ Substituting,

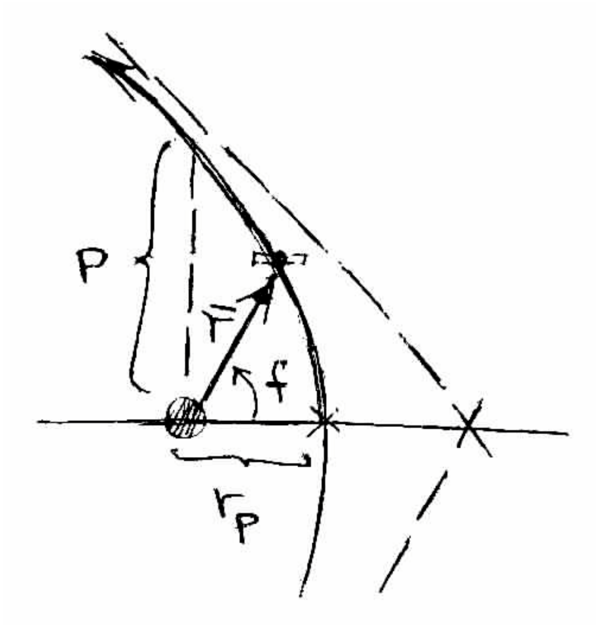
$$e = \sqrt{1 + \frac{2Eh^2}{\mu^2}}$$

Conic Sections

	V	**E**	**e**	**a**
Ellipse	$< \sqrt{2\mu / r}$	< 0	$0 \le e < 1$	> 0
Parabola	$= \sqrt{2\mu / r}$	= 0	$e = 1$	∞
Hyperbola	$> \sqrt{2\mu / r}$	> 0	$e > 1$	< 0

Terminology for r_a and r_p

	Generic	**Earth**	**Sun**	**Moon**
r_a	Apoapse	Apogee	Aphelion	Aposelenium
r_p	Periapse	Perigee	Perihelion	Periselenium



Escape Trajectory

Consider a spacecraft on a parabolic trajectory, approaching an asymptote as $r \to \infty$ The energy of the orbit is

$$E = \frac{v^2}{2} - \frac{\mu}{r} \Rightarrow \frac{v_\infty^2}{2} \text{ as } r \to \infty$$

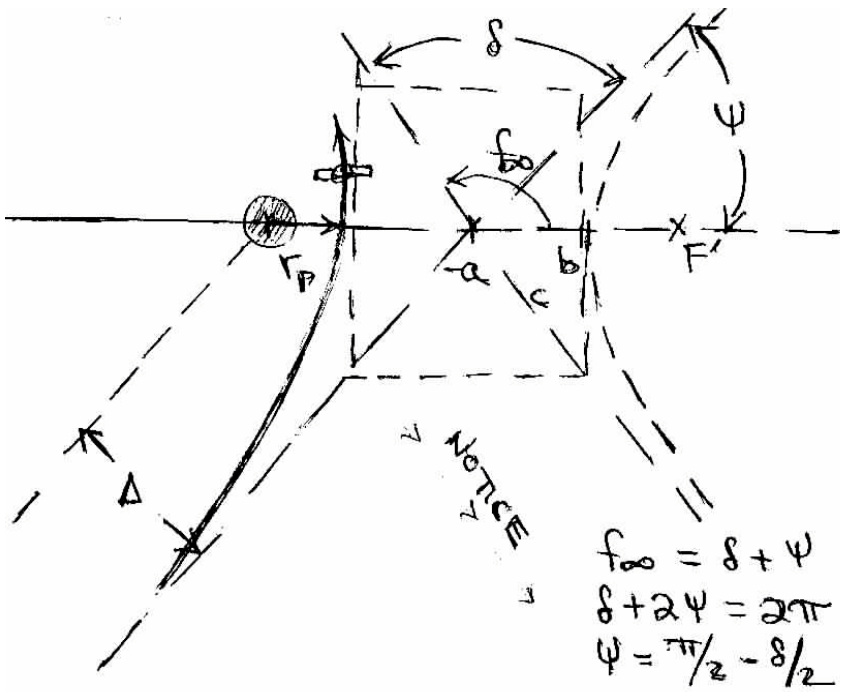
Substituting $E = -\frac{\mu}{2a}$ and solving

$$\boxed{v_\infty = -\frac{\mu}{a}}$$

This is formally called the hyperbolic excess velocity. It is an important design criteria for planet approach trajectories because v-infinity is also the velocity of a spacecraft as it approaches the planet far out on the other asymptote.

Turning Angle

Most exploration probes going to the outer planets make a close approach to Jupiter to get a boost in speed from that planet's huge mass. In designing the trajectory, you know the flight path going in and also the target path on the far side of Jupiter, so you need to design a fly by at just the right distance from Jupiter to put the spacecraft on the right path after fly by (i.e. along the asymptote)



Applying the trajectory equation (derived in the next section)

$$r = \frac{p}{1+e\cos f} \Rightarrow \frac{p}{r} = 1 + e\cos f = 0 \quad \text{as} \quad r \to \infty$$

Therefore, $\qquad e\cos f = -1$

Substituting $$\cos f_\infty = \cos\left(\frac{\pi}{2}+\frac{\delta}{2}\right) = -\sin\left(\frac{\delta}{2}\right)$$

Thus,

$$\boxed{\sin\left(\frac{\delta}{2}\right) = \frac{1}{e}}$$

The mission profile determines $\frac{\delta}{2}$, the turning angle and the above equation gives you eccentricity e. The s/c velocity on the incoming trajectory is known, and in this hyperbolic approach problem it's v_∞. Finally, knowing μ for Jupiter, you can solve for a. From this, the closest approach $r_p = a(1-e)$

Another convention is to specify Δ the distance between an asymptote and a parallel line through the target planet.

$$\boxed{h = r_p v_p = \Delta v_\infty}$$

And

$$e = 1 + \frac{r_p v_\infty^2}{\mu}$$

This last equation comes from solving the trajectory equation at periapse and applying the energy relation for the semi major axis of passage.

$$r_p = \frac{p}{1-e\cos f} = \frac{a\left(1-e^2\right)}{1+e\cos f}$$

Where,

$$r_p = \frac{a\left(1-e^2\right)}{1+e} = a\left(1-e\right)$$

And,

$$v_\infty = -\frac{\mu}{a} \quad => \quad a = -\frac{\mu}{v_\infty^2}$$

Thus,

$$r_p = -\frac{\mu}{v_\infty^2}(1-e)$$

Rearranging,

$$\boxed{e = 1 + \frac{r_p v_\infty^2}{\mu}}$$

Empty Focus

A complete hyperbola has two parabolic shaped curves, oriented along the asymptotes and foci as indicated in a previous figure. Observe that

$$c^2 = a^2 + b^2$$

If you know a, and e = c/a gives you c. Then you can solve for b, and then find r_p in the figure.

The opposite "parabola" around the empty focus, by the way, is the path that a charged particle would take on approaching a similarly charged particle on the subatomic level ~ versus the planetary type approach, between particles of opposite charge (i.e. gravity exhibits a force of attraction; two particles on the atomic level with the same charge repel each other). The phenomena is exactly the same because the force between charged particles is, like gravity, an inverse squared force ~ only the constant of proportionality differs; the geometry is the same.

Later the Three Body Problem will be developed in detail ~ two massive primaries and a much smaller third body moving around them. There is a whole body of research in quantum chemistry on the subatomic 3BP. This is a unique situation in physics ~ what you learn about the astronomical 3BP from analysis and observations of celestial bodies, translates immediately to a better understanding of subatomic phenomena and forces.

Conic Sections

The previous analysis has used the fact that the following equations are valid for all conic sections - ellipses, parabolas, and hyperbolas.

$$r = \frac{p}{1 + e\cos f}$$

$$r_p = a(1 - e)$$

$$p = a\left(1 - e^2\right)$$

$$x = r\cos f \text{ and } y = r\sin f$$

$$E = \frac{v^2}{2} - \frac{\mu}{r} = -\frac{\mu}{2a}$$

$$M = n\left(t - t_p\right)$$

(the first and last equations will be derived shortly)

Velocity Hodograph

A common part of a dynamic systems analysis is to investigate motion in terms of velocity ~ i.e. in the $\dot{x}\dot{y}$ plane instead of the customary xy plane. Consider the radius vector of a spacecraft in the orbital plane.

$$\vec{r} = \begin{bmatrix} r\cos f \\ r\sin f \\ 0 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Starting with the trajectory equation,

$$r = \frac{p}{1+e\cos f} \text{ where } r^2\dot{f} = h$$

$$\dot{r} = \frac{r\dot{f}\sin f}{1+e\cos f} = \sqrt{\frac{\mu}{p}}(e\cos f)$$

And,

$$r\dot{f} = \frac{h}{r} = \frac{\sqrt{\mu p}\,(1+e\cos f)}{p} = \sqrt{\frac{\mu}{p}}(1+e\cos f)$$

Substituting,

$$\vec{v} = \sqrt{\frac{\mu}{p}}\begin{bmatrix} -\sin f \\ e+\cos f \\ 0 \end{bmatrix}$$

Now observe that

$$\dot{y} - e\sqrt{\frac{\mu}{p}} = \sqrt{\frac{\mu}{p}}\cos f$$

And recognizing that the $\dot{x}$ and $\dot{y}$ terms have $\sqrt{\mu/p}$ you get from $\left(\cos^2+\sin^2=1\right)\sqrt{\mu/p}$,

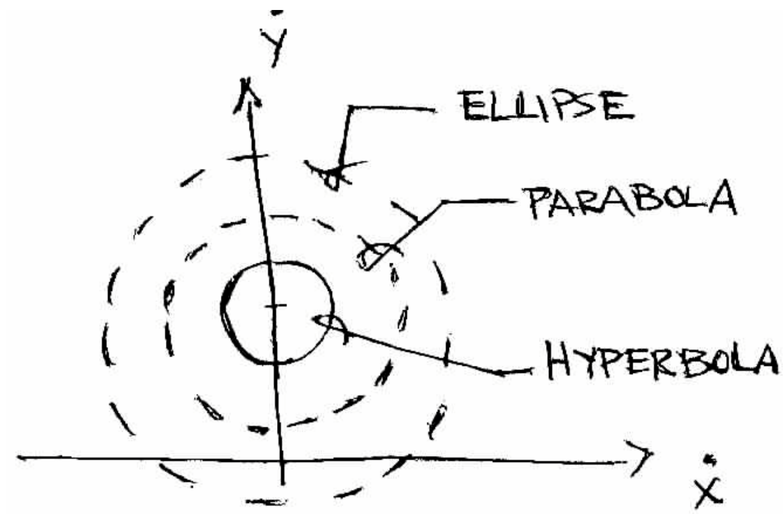
$$\dot{x}^2 + \left(\dot{y} - e\sqrt{\frac{\mu}{p}}\right)^2 = \sqrt{\frac{\mu}{p}}$$

which is the form of a circle with center at (x_0, y_0)

$$(x-x_0)^2 + (y-y_0)^2 = Radius$$

Thus, the graph is a circle with center at $\left(0, e\sqrt{\mu/p}\right)$, radius = $\sqrt{\mu/p}$

Finally, since these equations in f are valid for all conic sections, you find in the hodograph plane that the parabola and hyperbola are circles with center at $\left(0, e\sqrt{\mu/p}\right)$



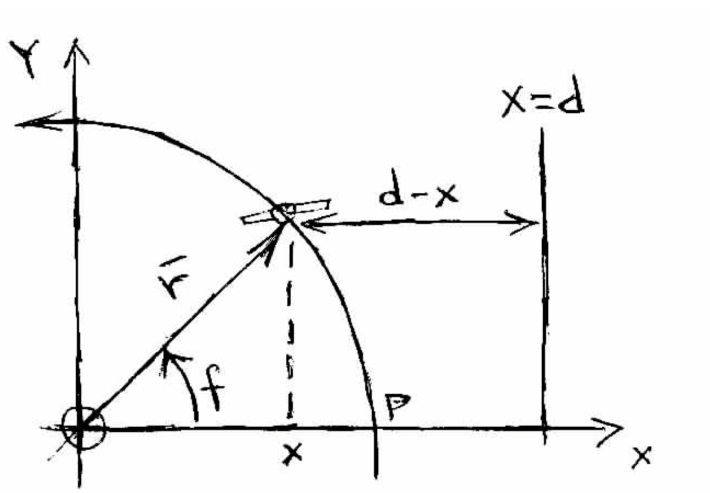
IV. The Trajectory Equation

TOPICS: Conic Sections
Simple Harmonic Motion
Argument of Periapse
Poincare's Relativity

Conic Sections

A useful formula in the theory of orbits called the Trajectory Equation will first be derived using basic concepts of geometry. The same equation is then derived from a more rigorous approach using classical dynamics and vector algebra. A comparison of the two equation provides valuable insights into conic sections, the parameter p, and the orientation of the orbit in the celestial sphere.

A simple derivation of the trajectory equation is found in most elementary calculus textbooks. It involves a fixed point F, the focus, and a fixed line called the directrix.



The geometry is described by the ratio e, the eccentricity, as follows

$$\frac{|PF|}{|PD|} = \frac{r}{d-x} = e$$

where ellipse 0 < e < 1
parabola e = 1
hyperbola e > 1

The semi parameter p is easily recognized from

$$e = \frac{p}{d} \quad \text{=>} \quad p = ed$$

From the illustration, observe that $x = r\cos f$. Substituting,

$$r = ed - ex = p\left(r\cos f\right)$$

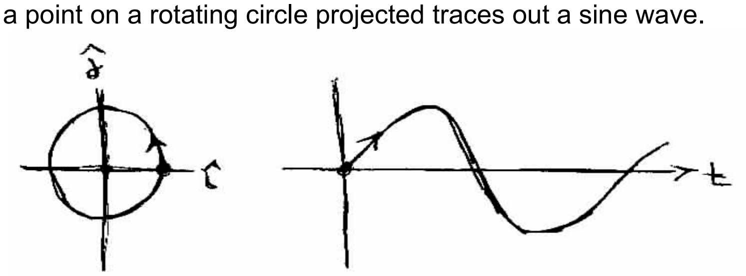
Thus,

$$\boxed{r = \frac{p}{1 + e\cos f}}$$

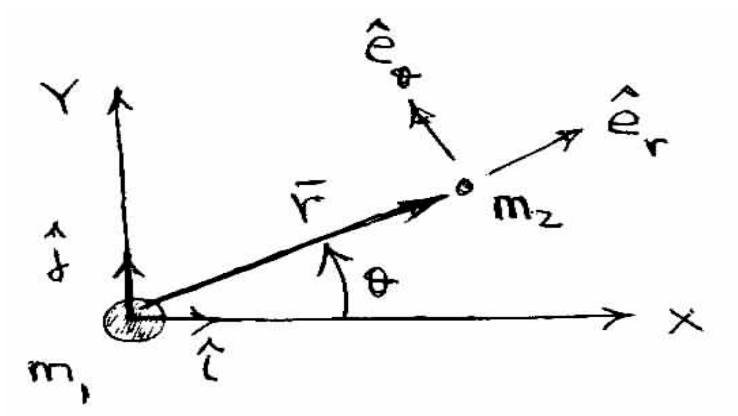
This formula is valid for all conic sections. Now to develop the same equation using the more exacting methods of classical dynamics.

Simple Harmonic Oscillation

A fundamental relationship between orbital motion and linear motion is that a point on a rotating circle projected traces out a sine wave.



Consider the Two Body Problem in the following coordinate system set up in the plane of the orbit, with unit radial and tangential vectors at the orbiting mass.



Define the vectors at the orbiting mass in terms of the inertial reference frame, as follows.

$$\hat{e}_\theta = -\sin\theta\hat{i} + \cos\theta\hat{j}$$

from which $\quad \dot{\hat{e}}_\theta = \dot{\theta}\hat{e}_\theta$

$$\hat{e}_r = \cos\theta\hat{i} + \sin\theta\hat{j}$$

from which $\quad \dot{\hat{e}}_r = -\dot{\theta}\hat{e}_r$

Now consider,

$$\vec{r} = r\hat{e}_r$$

$$\dot{\vec{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$$

$$\ddot{\vec{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{e}_r = -\frac{\mu}{r^2}\hat{e}_r$$

Now separate the $\hat{e}_r$ and $\hat{e}_\theta$ terms, and form two coupled equations

(1) $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ => $\frac{d}{dt}\left(r^2\dot{\theta}\right) = 0$

Thus, $\boxed{r^2\dot{\theta} = h}$ a useful formula

(2) $\ddot{r} - r\dot{\theta}^2 = -\frac{\mu}{r^2}$

This equation can be reduced to a linear differential equation by making two substitutions:

1. change the independent variable from t to θ
2. change the dependent variable from r to 1/s

From the chain rule we know that the first substitution requires:

$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt}$ and generally; $\dot{\theta} = \frac{h}{r^2}$

And so, from the first substitution

$$\frac{d}{dt} = \frac{h}{r^2}\frac{d}{d\theta} = hu^2\frac{d}{d\theta}$$

Now determine the velocity $\dot{r}$ and the acceleration $\ddot{r}$ (both scalar)

$$\dot{r} = \frac{dr}{dt}$$

$$\ddot{r} = hu^2\frac{d}{d\theta}\left(\frac{dr}{dt}\right)$$

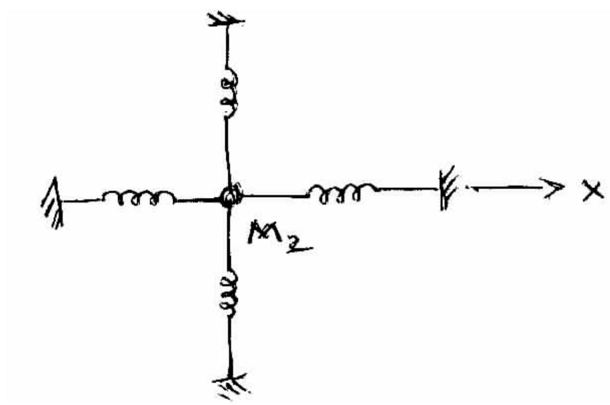
So,

$$r\dot{\theta}^2 = \frac{h^2}{r^3} = h^2u^3$$

Substituting back into the original equation,

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} = \text{constant}$$

Which is an equation for a simple harmonic oscillator, an undamped spring pass system.



The solution to a linear second order differential equation is

$$\ddot{u} + u = 0 \quad => \quad u_h = A\cos(\theta - \omega)$$

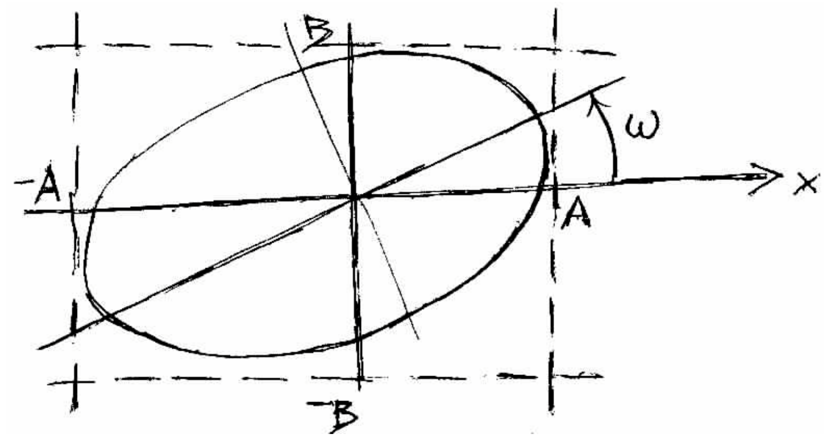
The particular solution is just $u_p = \mu / h^2$ and the general solution is just $u_h + u_p$ therefore

$$u = \frac{\mu}{h^2} + A\cos(\theta - \omega)$$

where θ, ω, and A are as yet undetermined constants.

The motion of the mass in the middle of the above spring system describes an ellipse according to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



Substituting back into the original equation $r = 1/u$ you get

$$r=\frac{1}{u}=\frac{h^2/\mu}{1+\frac{Ah^2}{\mu}\cos(\theta-\omega)}$$

Comparing this equation to the geometric solution you find that

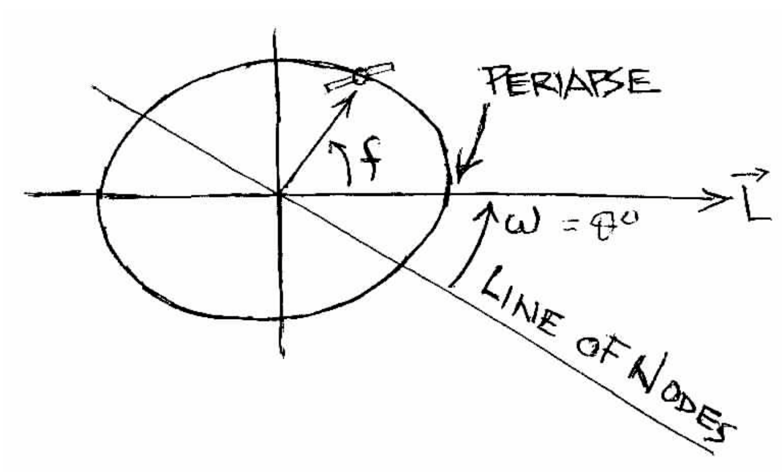
$f=\theta-\omega$	true anomaly
$p=h^2/\mu$	semi parameter
$e=Ah^2/\mu$	eccentricity

Substituting, you get the trajectory equation

$$r = \frac{p}{1 + e\cos f}$$

Argument of Periapse

The term ω in the above analysis comes about because this was a generalized solution - the geometric solution by aligning the conic section so periapse was on the x-axis.



In a generalized solution, the orbit is not in the plane of the paper, but at some angular inclination. The orbital plane and celestial plane (i.e. the plane of the page) intersect along the line of nodes. By convention, orbital motion is counterclockwise, so for the cosine term in the trajectory equation to be aligned from periapse, the value is adjusted by the argument of periapse.

The important function was developed from the above derivation

$$\boxed{p = \frac{h^2}{\mu}}$$

To further characterize p, consider

$$f = 0 \quad \Rightarrow \quad r_p = \frac{p}{1+e}$$

$$f = \pi \quad \Rightarrow \quad r_a = \frac{p}{1-e}$$

Combine these by substituting into $r_p + r_a = 2a$

$$\frac{p}{1+e} + \frac{p}{1-e} = \frac{2p}{1-e^2} = 2a$$

And so,

$$\boxed{p = a\left(1-e^2\right)}$$

To relate the constant A to the geometry of the problem, consider the equation for p,

$$p = \frac{h^2}{\mu} = a\left(1-e^2\right)$$

Leaving it to the reader to produce A from $e = A\frac{h^2}{\mu}$ in the form of the derivation.

The Eccentricity Vector

Strictly speaking, the line of nodes and periapse have no formal identification in dynamics, except as shown in the figures. The line of notes usually coincides with the x-axis. In orbital mechanics, the periapse is by convention on the x-axis (r_p is at $f = 0$). The following derivation indicates this is more than a convention.

An important vector constant was discovered by Laplace called the eccentricity vector.

(1) $$\vec{L} = \dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{r} = \dot{\vec{r}} \times \vec{h} - \mu \hat{r}$$

where $\hat{r}$ is a vector of unit length. Taking the time derivative of $\vec{L}$

$$\dot{\vec{L}} = \ddot{\vec{r}} \times \vec{h} - \mu \dot{\hat{r}}$$

$$= -\mu \frac{\vec{r}}{r^3} \times \vec{h} - \mu \frac{\vec{h} \times \vec{r}}{r^3} = 0$$

Thus $\vec{L}$ is constant in magnitude. Furthermore it is clear that $\vec{L} \times \vec{h} = 0$, so the two vectors are not independent. From the algebra,

$$\vec{L} \bullet \vec{r} = h - \mu r = Lr\cos\theta$$

where θ is the angle between $\vec{L}$ and $\vec{r}$. Solving for r,

(2) $$r = \frac{\dfrac{h^2}{\mu}}{1 + \left(\dfrac{L}{\mu}\right)\cos\theta} = \frac{p}{1 + e\cos f}$$

Comparing this to equation (2.7), it is easy to see that θ is the true anomaly. Thus, $\vec{L}$ points toward periapse and

(3) $$\vec{e} = \frac{\vec{L}}{\mu} \text{ and so, } e = \frac{L}{\mu}$$

Thus the name "eccentricity vector," since the length of $\vec{L}$ is the magnitude of the eccentricity of the ellipse.

This derivation implies there is a preferred orientation of an ellipse, and that the rule - not the exception - is an angle to the line of nodes. Having derived the trajectory equation from $\vec{L}$ shows the eccentricity vector applies to all conic sections.

Mathematical Physics

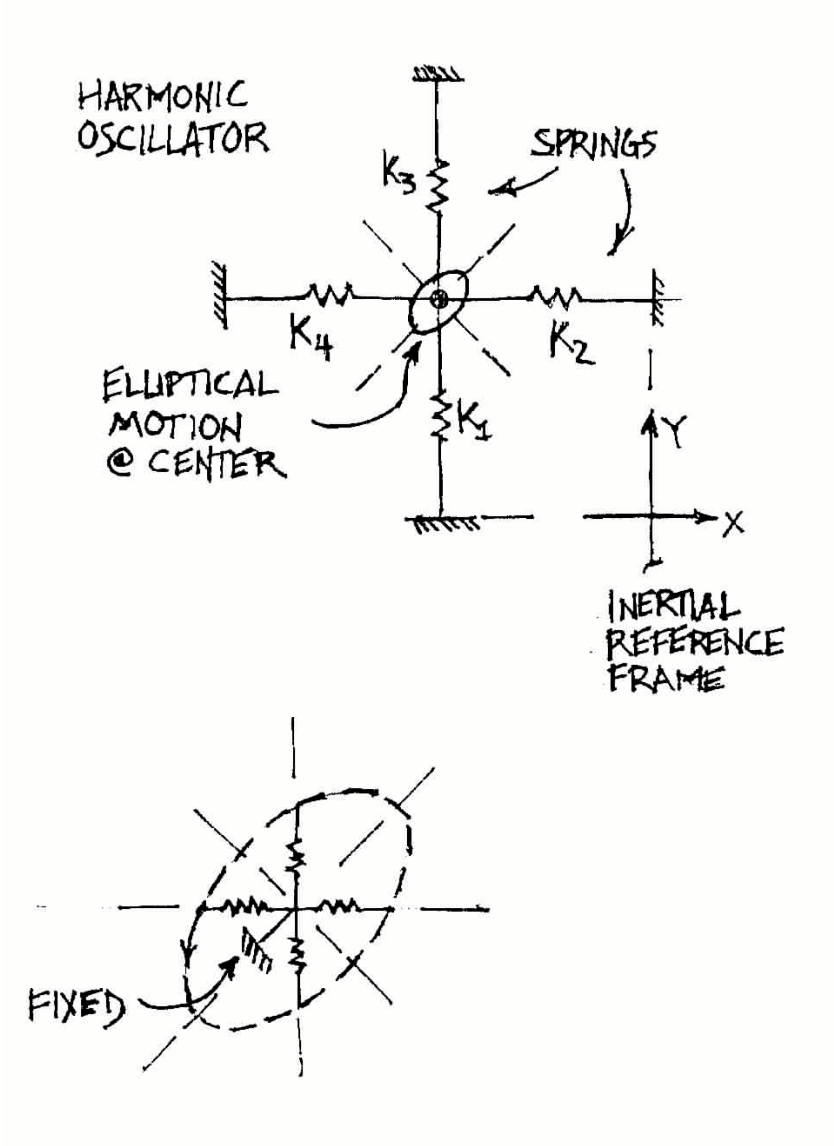
It's been said that when scientists study natural phenomena in order to devise a mathematical model for the data, they use the branch of math that is most familiar to them. Kepler, in trying to fit a model to Tycho Brahe's data, used geometry ~ as did Newton. The next generation of scientists used calculus, and so on until the modern era when statistics are used to model data in virtually every science from psychology to physics.

This book is written to offer the most clear grasp of gravity to the student, and naturally tends toward geometric modeling because that is the most fundamental way to show things and the simplest to visualize and comprehend. As a consequence, geometry will be the mathematics of choice in modeling inexplicable phenomena.

Upon further consideration, the presence of this out-of-plane angle that is called the argument of periapse is quite an unexpected result. The problem was set up in the xy plane, motion and forces were defined in the xy plane, and then the result implies that motion limited to the xy plane is an exception - perhaps even unstable. (Note: a 3D simple harmonic oscillator has a third set of springs along the z-axis, as illustrated on the following page.)

In real life everybody knows that motion happens in three dimensions, and so this projection of a 2D mathematical system into 3D is so expected that it goes unnoticed. However, the mathematics is not inherently 3D and so when the most simple dynamic model in 2D automatically extends itself into 3D ~ well, it's a nice and convenient thing to happen, but alarms go off at the same time. It's like finding out that a Focoult Pendulum - whose plane of motion rotates with the rotation of Earth - this problem says that mathematically a simple harmonic motion will adjust is plane even if the Earth did not rotate; or, at least, saying there is a built in tendency to do so.

Not to make much ado about nothing, this anomaly is formally recognized her simply to suggest the presence of subtle forces not yet quantifiable. The h/area notion, which similarly suggested a pattern in the motion with regards to the orbital plane, is noticed again. The whole idea is reinforced, if not validated.



The motivation for this exercise in logic is that we know such a force exists, from Relativity Theory. The main three proofs of Relativity all involve a subtle rotational force ~ bending of light as it passes close to the sun, the slow rotation of Mercury's line of nodes, and an infinitesimal time differential in atomic clocks orbiting Earth at high speeds.

The reward, of course, would be to find a simple explanation for Relativistic phenomena - if not an actual idea worthy of the stature of a physical law, then perhaps an easy way to incorporate Relativistic affects in a simple model to make it that much more accurate ~ without adopting the arcane complexity of tensor algebra and time/space Lorenz transformations.

Consider also that even a slightly inaccurate geometric model would seriously undermine Relativity. A new model gets many bonus points just for simplicity. Copernicus' simple heliocentric model was widely accepted over the ultra complex geocentric model of Ptolemy, despite a few errors that required small epicycles and cycles. Scientists like simplicity because nature trends toward simplicity always.

Poincare's Relativity

Few people know that the French mathematician and celestial mechanics worker, Henri Poincare developed the Theory of Relativity thirty years before Einstein. Poincare and Einstein arrived at the exact same results ~ the only difference is that Einstein's derivation was based on electromagnetic theory and Poincare's on celestial mechanics theory. So it is not unreasonable to expect to find the subtle machinations of Relativity in the fundamental theory of orbits.

You are aware of the Two Body Problem and the Three Body Problem. Another common problem is the N Body Problem or Many Body Problem. The equations of the N Body Problem reduce directly to the same equations as for incompressible flow in fluid dynamics. Space behaves just like a fluid when Many Bodies are acting on one small mass. This fits in well with Poincare's derivation of Relativity - that there is an aether in space, was his only assumption.

It's well known now that there's no aether in space. Space is an absolute vacuum. This means that Poincare's Relativity Theory is based on a false assumption, and the conclusions are not correct. Moreover, it means that if you derive Einstein's equations backward along Poincare's method, then Einstein's Theory or Relativity requires the existence of an aether ~ and so it too is incorrect.

That's what the mathematics they used says. It's what the geometry and (so far) the math suggest, with the exception of the train of thought in this account. Celestial mechanics has always had alternative Newtonian explanations for the phenomena explained by Relativity. These explanations will be described in due course, and refined to fit these new insights ~ resulting in a new mechanistic explanation for the three major tests of Relativity.

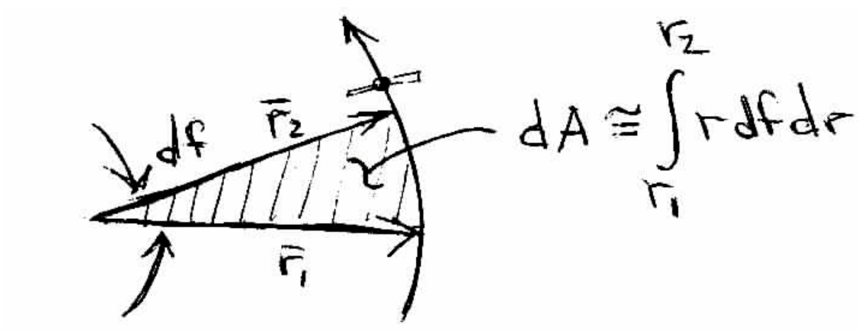
Actually, there is one way to assimilate Einstein's and Poincare's Relativity equitably. That is to say, yes, there is an "aether" at Newtonian speeds, but that space behaves like a fluid at speeds approaching the speed of light ~ however, even though Relativity implies all motion stops at c, it really just slows down dramatically ~ like going from air into water.

V. 2nd Body in Motion

TOPICS: The Law of Equal Areas
Kepler's Third Law
Orbital Elements

Kepler's Second Law

Kepler's Law of Equal Areas says a line from the central body at one focus of the ellipse to the orbiting planet sweeps out equal areas in equal times. The geometry of the problem is:



Assuming $r_1 \cong r_2$ (i.e. as $df \to 0$),

$$dA = \tfrac{1}{2} r^2 df$$

$$\frac{dA}{dt} = \tfrac{1}{2} r^2 \frac{df}{dt}$$

Recall $h = r^2 \dot{f} = r^2 \dfrac{df}{dt}$. Substituting, $\dfrac{dA}{dt} = \tfrac{1}{2} h =$ constant

Meaning, a line between the focus and a planet sweeps out equal areas in equal times.

Kepler's Third Law is a simple consequence of the second. Rearranging the last equation,

$$2dA = hdt$$

Integrating over the complete period of the orbit, τ

$$2\int dA = h\int dt$$

Integrating dt from 0 to τ, the complete period of the orbit. The area of an ellipse equals πab, so

$$\tau = \frac{2\pi ab}{h}$$

Where $b = a\sqrt{1-e^2}$ and $b^2 = a^2\left(1-e^2\right) = ap$ As a result,

$$\boxed{h = \sqrt{\mu p}}$$

Substituting $p = \sqrt{a\left(1-e^2\right)}$ you end up with

$$\tau = \frac{2\pi a^{3/2} p^{1/2}}{\mu^{1/2}\sqrt{a\left(1-e^2\right)}} = 2\pi\sqrt{\frac{a^3}{\mu}}$$

Consequently, the square of the period is proportional to the cube of the semi major axis. If you define the mean motion, n, as

$$n = \frac{2\pi}{\tau} rad/\sec$$

Then you get the important relation

$$\boxed{\mu = n^2 a^3} \text{ or } n = \sqrt{\frac{\mu}{a^3}}$$

Where n is the mean angular rate of change of an object in a simple Two Body orbit.

Mathematical Physics (again)

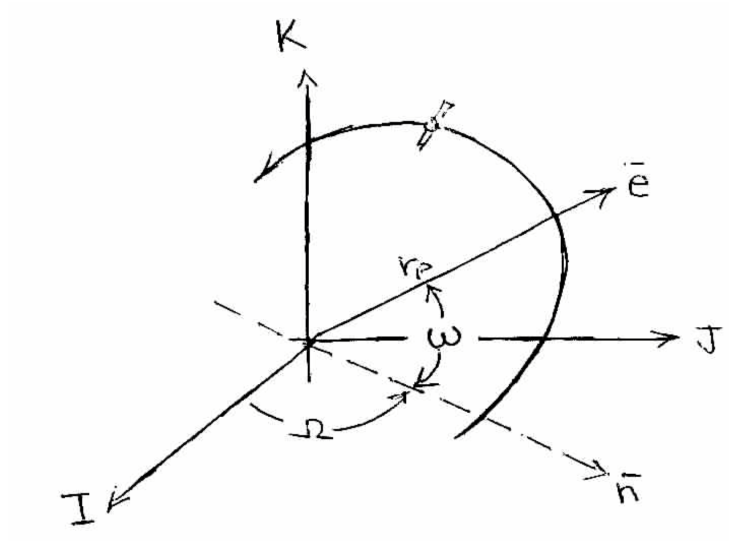
Observe how precisely this derivation of Kepler's 2nd Law fits into the overall Two Body Theory. It's perfect! If you were to assume $r_1 \neq r_2$ for elliptical orbits and derive the result, it would be very messy ~ and it wouldn't fit in with the rest of the theory so seamlessly. Still, the fact remains that $r_1 \neq r_2$ for elliptical orbits, and assuming $r_1 = r_2$ translates into an error that is consistent with previous remarks on $\left|\vec{h}\right|$, its area as a cross product, and its association with the geometric parameters of the ellipse itself (i.e. a and b). A function for the rate of change of r will be derived in the next section

$$\dot{r} = \frac{he\sin f}{p}$$

which shows that $\dot{r}$ varies directly as $\sin f$ (everything else is constant) so the greatest discrepancy between r_1 and r_2 is when $f = 90^{\circ}$ or $f = 290^{\circ}$. These points are along the axis of the semi parameter and $\dot{r} = 0$ along the eccentricity vector, which is perpendicular to p.

Orbital Elements

The orbit can be described in a right handed coordinate system called IJK coordinates. The vector **I** is fixed in inertial space, directed to the First Point or Aries; **J** is at an angle of ninety degrees such that the **IJ** plane is the Earth's equatorial plane; and **K** is toward the North pole.



The orbital plane and the **IJ** plane intersect along the vector **n,** the line of nodes; with the orbital plane rotated about this vector by an amount equal to the inclination of the orbit. The longitude of the ascending node of the orbit is measured in the Earth's equatorial plane as shown, and the argument of periapse is the angle measured in the plane of the orbit to the vector **e** through periapse. The symbols for these angles are

i = inclination of the orbit
ω = argument of the periapse
Ω = longitude of the ascending node

Position and velocity in the plane of the orbit can be calculated directly from the orbital elements.

$$\vec{r} = \frac{p}{1 + e\cos f}\begin{bmatrix} \cos f \\ \sin f \\ 0 \end{bmatrix}$$

$$\vec{v} = \sqrt{\frac{\mu}{p}} \begin{bmatrix} -\sin f \\ \cos f + e \\ 0 \end{bmatrix}$$

Given $\vec{r}$ and $\vec{v}$ in the plane of the orbit, the orientation in the **IJK** system can be determined by the following series of calculations

$$\vec{h} = \vec{r} \times \vec{v}$$

$$\vec{e} = \frac{1}{\mu}\left(\vec{v} \times \vec{h}\right) - \frac{\vec{r}}{r} \text{ where } e = \left|\vec{e}\right|$$

$$\vec{n} = \vec{k} \times \vec{h} \text{ where } \vec{k} = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$$

$$\cos i = \frac{\vec{k} \bullet \vec{h}}{kh} \text{ or } \cos i = \frac{h_z}{h} \text{ or } \sin i = n$$

$$\cos \Omega = \frac{\vec{I} \bullet \vec{n}}{In} \text{ and if } r_x < 0 \text{ then } \Omega = 360 - \Omega$$

$$\cos \omega = \frac{\vec{n} \bullet \vec{e}}{ne} \text{ and if } e_z < 0 \text{ then } \omega = 360 - \omega$$

$$\cos f = \frac{\vec{e} \bullet \vec{r}}{er} \text{ and if } \vec{r} \bullet \vec{v} < 0 \text{ then } f = 360 - f$$

Translating coordinates from **PQW** to **IJK** is done by a series of three rotations about the respective axes. These rotations must be done in the order specified from right to left.

$$r_{IJK} = ROT3\left(-\Omega\right) ROT1\left(-i\right) ROT3\left(-\omega\right) \vec{r}_{PQW}$$

where the rotations about the x- y- and z-axes in a right handed coordinate system are represented by the following matrices

$$ROT1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos & \sin \\ 0 & -\sin & \cos \end{bmatrix}$$

$$ROT2 = \begin{bmatrix} \cos & 0 & -\sin \\ 0 & 1 & 0 \\ \sin & 0 & \cos \end{bmatrix}$$

$$ROT3 = \begin{bmatrix} \cos & \sin & 0 \\ -\sin & \cos & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It is interesting to notice that

$$r_{IJK} = \begin{bmatrix} \hat{P} & \hat{Q} & \vec{W} \end{bmatrix} \vec{r}_{PQW}$$

where the column vectors of the single 3x3 transformation matrix from **PQW** to **IJK** in equation (10) are unit vectors in the orbital plane. That is, the vector **P** configured in the orbital plane points in the direction of periapse (i.e. the eccentricity vector); the vector **Q** in the orbital plane is at a true anomaly of ninety degrees towards the semi-perimeter; and the vector **W** in the orbital plane is perpendicular to the plane of the orbit, directed upwards. These three vectors form the axes of the **PQW** coordinate system, which is the formal name of the right hand oriented inertial reference frame used in identifying the orbital elements of a, e, p, n, f, M, and E in previous sections.

Latitude and longitude are found by rotating the **IJK** coordinates through another angle, the angle equivalent of Greenwich Standard Time (GST) of the site

$$r_{ECEF} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix} = ROT3(\theta_{GST})\vec{r}_{IJK}$$

longitude, $\lambda = \arctan\left(\frac{r_y}{r_x}\right)$

latitude, $\phi = \arctan\left(\frac{r_z}{r}\right)$

Then, azimuth and elevation are calculated from the range vector, as follows.

$$\vec{\rho}_{IJK} = \vec{r}_{IJK} - \vec{r}_{site,IJK}$$

$$\rho_{SEZ} = ROT2(90° - \phi)\, ROT3(\lambda)\, ROT3(\theta_{GST})\, \vec{\rho}_{IJK}$$

$$= \rho \begin{bmatrix} -\cos(el)\cos(Az) \\ \cos(el)\sin(Az) \\ \sin(el) \end{bmatrix} = \begin{bmatrix} \rho_x \\ \rho_y \\ \rho_z \end{bmatrix}$$

Where,

$$Az = \arctan\left(-\frac{\rho_y}{\rho_x}\right)$$

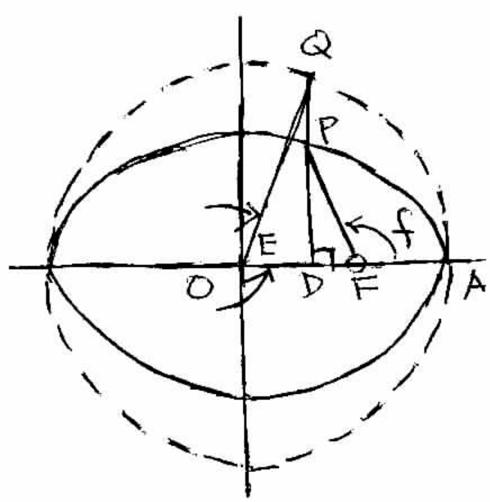
$$El = \arcsin\left(\frac{\rho_z}{\rho}\right)$$

VI Kepler's Equation

TOPICS: Graphical Derivation
Rigorous Derivation
Solving Kepler's Equation
Kepler's Problem

Graphical Derivation

This section derives an important function called Kepler's Equation, then shows how to solve it iteratively. The derivation is in two parts ~ a geometric proof and a rigorous mathematical proof. The geometric derivation establishes the parameters of the ellipse (i.e. the mean anomaly) versus Kepler's equation and the second derivation develops two important relationships en route to the same solution. This is exactly t he method used to study the Trajectory Equation: both implying there is a subtle geometric structure separate from the straight forward mathematical development (as with the Law of Equal Areas, and even Netwon's Law of Gravitation). What began as a mere curiosity with h and the cross product area has become a trend ~ if not the norm.



An analytic expression for the mean anomaly can be found from the geometry of the ellipse versus the eccentric anomaly.

$$\frac{M}{2\pi} = \frac{area \sim AFP}{area \sim ellipse} = \frac{area \sim AFQ}{area \sim circle}$$

$$AFQ = ACQ - FCQ$$

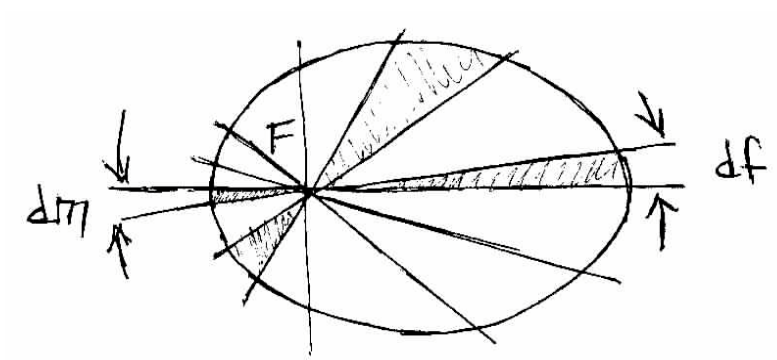
$$= ½a^2 E - ½ae\left(a \sin E\right)$$

$$\frac{M}{2\pi} = \frac{a^2\left(E - e\sin E\right)}{2\pi a^2}$$

Which reduces to Kepler's Equation.

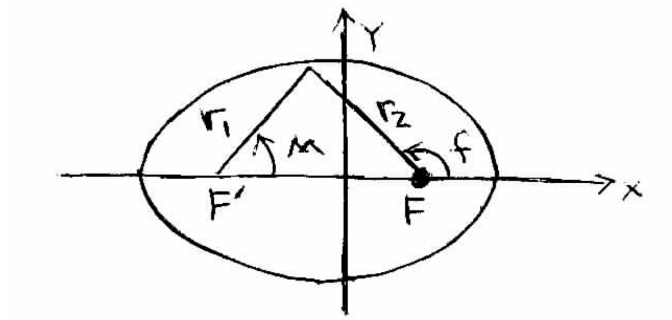
$$\boxed{M = E - e\sin E}$$

The mean anomaly M is an angle that changes at a constant rate as the planet travels in its elliptical orbit. The mean anomaly has no geometric equivalent like E; however, it is closely approximated for small eccentric orbits by a line from the empty focus to the planet.



As eccentricity increases, adding a small epicycle at M increases the accuracy of the model.

Recall from the geometry of the ellipse,



$$r_1 + r_2 = 2a$$

$$r_1 \sin M = r_2 \sin(\pi - f) = r_2 \cos f$$

$$(a - r)\sin M = r \cos f$$

This last equation suggests a "coordinating influence" from the empty focus. Earlier comments implied a complimentary force at the center of the ellipse; this here is a second such subliminal force.

Earlier a correspondence was hinted between area and the angular momentum vector ~ this derivation for the main anomaly shows a direct correlation between area and the mathematics.

Rigorous Derivation

Consider a coordinate system centered at the focus occupied by the central body. It is clear that

$$\boxed{x = a\cos E - ae}$$

Where ae is the distance between the center of the ellipse and each focus. From the equation for an ellipse,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \Rightarrow \quad \frac{y_{ellipse}}{y_{circle}} = \frac{b}{a}$$

Therefore,

$$y_{ellipse} = \frac{b}{a}\left(y_{circle}\right) = \frac{b}{a}\left(a\sin E\right) \quad => \quad \boxed{y = b\sin E}$$

Which is a remarkable result if you think about it ~ the y-coordinate following a point on a circle of radius b.

Note how the x- and y-coordinates are decoupled, each depending on the independent constants a and b, the semi major axis, and the semi minor axis.

$$r^2 = x^2 + y^2$$

$$= \left(a\cos E - ae\right)^2 + \left(b\sin E\right)^2 \text{ where } b = a\sqrt{1-e^2}$$

$$= a^2\left[\left(\cos E - e\right)^2 + \left(\sqrt{1-e^2}\sin E\right)^2\right]$$

$$= a^2\left[\cos^2 E + \sin^2 E + e^2 - 2e\cos E - e^2\sin^2 E\right]$$

$$= a^2\left[1 + (e^2\sin^2 E) - 2e\cos E\right]$$

$$= a^2\left[1 + e^2\cos^2 E - 2e\cos E\right]$$

$$= a^2\left(1 - e\cos E\right)^2$$

$$\boxed{r = a\left(1 - e\cos E\right)}$$

If you take time derivatives,

$$\dot{r} = ae\dot{E}\sin E$$

Now consider the trajectory equation,

$$r = \frac{p}{1 + e\cos f}$$

$$\dot{r} = \frac{-p\left(-e\sin f\right)\dot{f}}{\left(1 + e\cos f\right)^2} = \frac{p\left(e\sin f\right)\dot{f}r^2}{p^2}$$

But $h = r^2\dot{f}$ and so,

$$\boxed{\dot{r} = \frac{he\sin f}{p}} = ae\dot{E}\sin E \text{ from above}$$

Recall, $y = r\sin f = b\sin E$

Noticing there is a $\sin f$ and $\sin E$ on opposite sides of the time derivative for r,

$$\begin{aligned} ae\sin\dot{E}\sin E &= \frac{he}{p}\sin f \\ &= \frac{ha}{p}\left[\frac{b}{r}\sin E\right] \end{aligned}$$

$$r\dot{E} = \frac{hb}{pa} \quad \text{now substituting,}$$

$$= \frac{\sqrt{\mu a\left(1-e^2\right)}a\sqrt{1-e^2}}{a\left(1-e^2\right)a} = \sqrt{\frac{\mu}{a}}$$

Collecting terms,

$$r\dot{E} = \sqrt{\frac{\mu}{a}}$$

Using the equation just derived for r in E,

$$a\left(1-e\cos E\right)\dot{E}=\sqrt{\frac{\mu}{a}}$$

$$\dot{E}-e\dot{E}\cos E=\frac{\mu^{1/2}}{a^{3/2}}=n$$

Integrating,

$$E-e\sin E=nt+\text{constant}$$

To find the constant, $E=0$ at periapse, where $t=t_p$ and so

$$E-e\sin E=n\left(t-t_p\right)=M$$

Where from the geometry of the first, graphical derivation, M is the mean anomaly. The mean and eccentric anomalies allow the orbit to be propagated through time.

A convenient relationship between the true and eccentric anomalies is

$$\tan\left(\frac{f}{2}\right)=\sqrt{\frac{1+e}{1-e}}\tan\left(\frac{E}{2}\right)$$

Otherwise, the mathematical derivation of M confirms the graphical derivation, and reinforces the notion of area as an integral function in the theory of orbits. It's interesting to note that the earlier idea of h and area takes on a new significance, as

$$\left|\vec{h}\right|=h=nab=\frac{2\pi ab}{\tau}$$

Where τ is the period and $2\pi ab$ is the area of an ellipse.

The following two cases illustrate just how the mean and eccentric anomalies can be used to propagate an orbit through time.

#1 if time is given; $M = n\left(t - t_p\right)$

(1) solve $M = E - e\sin E$ by interation

(2) $r = a\left(1 - e\cos E\right)$

(3) $r\cos f = a\left(\cos E - e\right)$ => solve for f

(4) $r\sin f = b\sin E$ => solve for f and quadrant

#2 if f is given,

(1) $r = \dfrac{p}{1 + e\cos f}$

(2) $a\cos E = r\cos f + ae$ => solve for E

(3) $b\sin E = r\sin f$ => solve for E and quadrant

(4) $E - e\sin E = n\left(t - t_p\right)$ => solve for t

The discussion has tried to convince the reader of the importance of geometry in the Two Body Problem. In these two examples, the steps to solve for the quadrants are not necessary if the geometry of the problem is known. In this regard, the geometry is actually an independent function of the system, a kind of envelope curve.

Solving Kepler's Equation

As you might expect, geometry can be used to solve Kepler's Equation directly. However, when using analytical methods, the solution must be found iteratively, using the calculus.

Consider a Taylor Series expansion for a function $f(\phi)$ where $\phi = x + \delta$ and $f(x)$ is known

$$f(\phi) = f(x) + f'(x)\delta + \frac{f''(x)\delta^2}{2!} + ...$$

Solving for δ,

$$\delta \cong -\frac{f(x)}{f'(x)}$$

This sets up a simple iteration,

$$x_{n+1} = x_n + \delta = x_n - \frac{f(x_n)}{f'(x_n)}$$

In the case of Kepler's Equation,

$$f(E) = E - e\sin E - M$$

$$\frac{f'(E)}{\delta E} = E - e\sin E - M$$

Thus,

$$\boxed{E_{n+1} = E_n + \frac{M - E_n + e\sin E_n}{1 - e\cos E_n}}$$

Usually $E_0 = M$ and so the interation goes like this:

$$E_2 = E_1 + \frac{M - E_1 + e\sin E_1}{1 - e\cos E_1}$$

$$E_1 = M + \frac{M - E_0 + e\sin E_0}{1 - e\cos E_0}$$

Until,

$$\left(E_{n+1} - E_n\right) \cong 0$$

A better first guess for E that saves many iterations is to use the first few terms of a series expansion in M

$$E = M + e\sin M + \tfrac{1}{2}e^2 \sin 2M \ldots$$

Kepler's Problem

It's interesting to note that Kepler's problem, even after he solved it, became the passion of all the mathematicians of the day. Famous scientists all over the world - Euler, LaPlace, D'Alembert, Gauss, Poisson, Hamilton, and Jacobi - all the giants of mathematics - they each (and dozens more) published their own *unique* solutions to Kepler's problem.

If you think about it, dozens of unique solutions to Kepler's Equation implies that there is something fundamental about the eccentric anomaly. It's more than a convenience, but a mathematical necessity integral to the orbit itself ~ like the line of nodes in the trajectory equation solution of simple harmonic motion.

The only unique thing about the eccentric anomaly is that it comes from the center of the ellipse and not from a focus. The analytic geometry shows that elliptical orbits are not unique to $1/r^2$ forces originating at the focus of a conic section. Elliptical orbits also happen for 1/r forces originating at the geometric center of the ellipse ~ a point which Kepler's equation has now endowed with extra importance.

VII. Applications

TOPICS: Ballistic Trajectories
Lambert's Problem

The theme so far has been the existence of some subtle inconsistencies in the Two Body Problem theory that hint of a relativistic phenomena taking place in orbital theory. The suppositions have focused on geometry, and the idea that there exists a geometric analogy to Relativity ~ as is the case with many of the derivations provided so far.

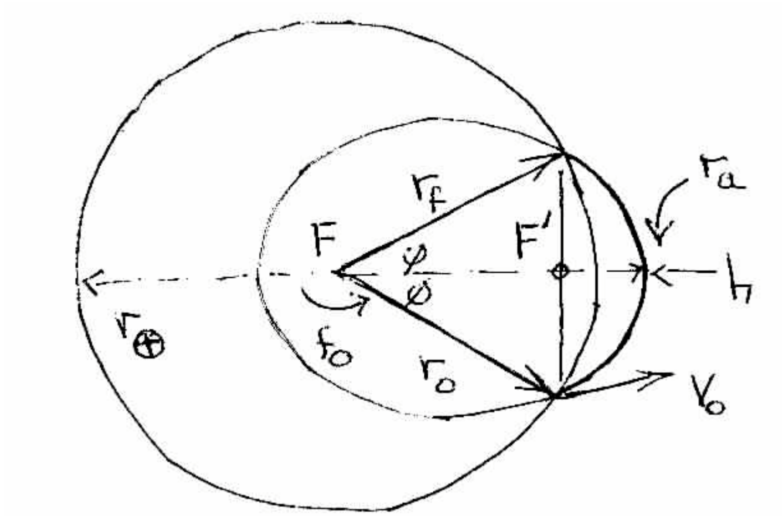
The geometric ideas posed so far, like hand written comments in the margins, are trending toward two separate quantities ~ functions of a and be relative to the center of the ellipse and the empty focus, respectively; and so forth. The evidence so far is strong, but inconclusive.

There remain three formal derivatives in the theory of Two Bodies ~ ballistic trajectories, lambert's problem, and (in the next section) the theory of f- and g-functions. The first two applications will now be developed and the results studied for any geometric anomalies.

Ballistic Trajectories

An important problem in orbital mechanics is the ballistic trajectory, a scenario where there are no thrusts applied after the initial burn. The most important practical application is for ICBM's, intercontinental ballistic missiles, with nuclear warheads.

The ICBM problem is set up with the center of mass of Earth at one focus of the ellipse, but in this case the ellipse is not as in the problems so far but such that most of the ellipse is inside the Earth itself, leaving just a portion of the elliptical flight path above the surface of Earth ~ the trajectory along which the missile moves. The problem assumes there is no drag or other forces, as happen close to Earth where the air is thick. Drag forces on a satellite are relatively consistent and act only to shorten the semi major axis, and that only after many orbits. Overall, the following model is remarkably accurate.



The following are evident from the geometry.

$$f_0 + f_f = 2\pi$$

$$E_0 + E_f = 2\pi$$

$$r_\oplus + h = r_a$$

$\Delta f = 2\phi$ = the range angle

As with the minimum energy Lambert problem, the optimal ballistic trajectory is when the chord between the initial and final position vectors goes through the empty focus of the ellipse.

$$2a = r_0 + r_f$$

because the minimum energy flight path has the smallest a, from

$$E = -\frac{\mu}{2a}$$

From the geometry, and symmetry, r_f is the semi-parameter

$$r_f = p = a(1-e^2) = r_{\oplus} \sin\phi$$

And combining functions,

$$2a = r_0 + p$$

$$a = r_{\oplus} \frac{1+\sin\phi}{2}$$

The true anomaly of the launch point f_0 can now be found from the trajectory equation and substituting for e

$$r_{\oplus} = \frac{a\left(1-e^2\right)}{1+e\cos f_0}$$

Kepler's equation can be used to determine the time of flight.

$$n\left(t_0 - t_p\right) = E_0 - e\sin E_0$$

$$-\left[n\left(t_f - t_p\right) = E_f - e\sin E_f\right]$$; adding (7a) and (7b)

$$\Delta t = \frac{2}{n}\left[\pi - E_0 + e\sin E_0\right]$$

An optimum eccentricity for the mission can now be calculated with a little effort.

$$r_0 = \frac{p}{1+e\cos f} \;;\quad p = a(1-e^2)$$

solving for a,

$$a = \frac{r_0(1+e\cos f_0)}{1-e^2}$$

Taking the derivative of this function for a with respect to e,

$$\frac{da}{de} = \frac{r_0\left(e^2\cos f_0 + 2e + \cos f_0\right)}{\left(1-e^2\right)^2} = 0$$

Using the quadratic formula to solve for e

$$e = \frac{-2 \pm \sqrt{4 - 4\cos^2 f_0}}{2\cos f_0} = \frac{-1 \pm \sin f_0}{\cos f_0}$$

Thus, the optimum eccentricity is the simple formula that is a function of the initial true anomaly only.

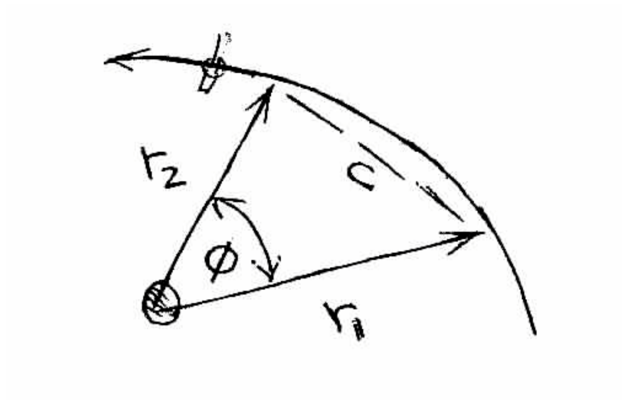
$$e = \frac{\sin f_0 - 1}{\cos f_0}$$

Generally, the ballistic trajectory analysis is symmetric, and in the set up is remarkable in that the focus is what in most other configurations is the empty focus - using the other focus the equations are far more complicated and convoluted. This confirms the idea that the empty focus may have more significance than is usually given to it.

Lambert's Problem

A common problem in mission planning is to find a flight path that goes between two known position vectors. The analysis gets its name from a theory proposed by Lambert that the transit time is a function of only the semi-major axis of the solution ellipse, the sum of the magnitudes of the position vectors, and the chord length.

There are many applications of this problem, e.g. when a spacecraft is to take other than a minimum energy Hohmann Transfer between to trajectories. It might be a rendezvous or the other spacecraft might be targeted for impact. There is a whole family of curves that include any given two radius vectors, but there is among these just one optimal path. It is also the simplest trajectory to find; that is, the minimum energy path between two position vectors.



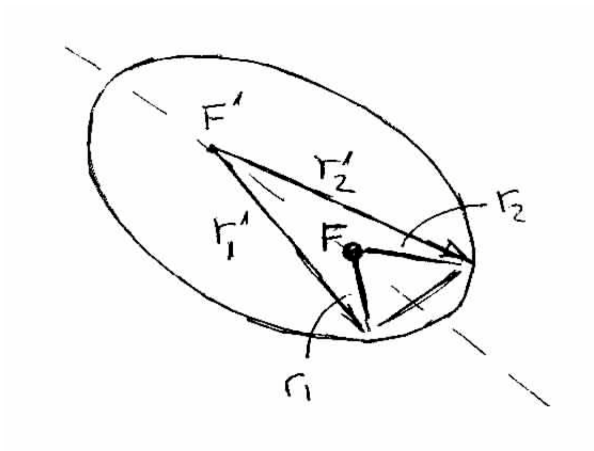
Several features are in evidence from the geometry and from the cosine law:

$$\cos\phi = \frac{\vec{r}_1 - \vec{r}_2}{r_1 r_2} \quad \text{where } \phi = \Delta f$$

$$c^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos\phi$$

Considering the two vectors in context of the minimum energy ellipse, it is possible to find the semi-major axis from the geometry of the problem.

You will note that the above derivation, unlike the derivation for the Law of Equal areas, does not assume $r_1 = r_2$ even though the trajectory is often on an elliptical path. Consequently, if you have doubted the consequences of that assumption for Kepler's Second Law, there is no reason to doubt it any longer.



The minimum energy transfer path is when the semi-major axis is the shortest.

$$E = -\frac{\mu}{2a}$$

The shortest semi-major axis is when the two vectors from the primary focus are co-linear; that is, if they coincide with the chord. By the geometry, the major axis is always equal to the length of the line from one focus to a point on the ellipse, then to the other focus

$$r_1^{'} + r_1 = 2a$$

$$r_2^{'} + r_2 = 2a$$

Adding equations (4) and solving for the minimum a,

$$a_{\min} = \frac{r_1 + r_2 + c}{4}$$

Another elegant formula exists for the eccentricity of this ellipse

$$e_{\min} = \frac{|r_2 - r_1|}{2}$$

The velocity vector at the first position vector is

$$\vec{v}_1 = \frac{h}{r_1 r_2 \sin\phi}\left\{\vec{r}_2 - \left[1 - \frac{r_2}{p}\langle 1 - \cos\phi\rangle\right]\vec{r}_1\right\}$$

The time of flight is found using the two Lambert angles

$$\cos\alpha = 1 - \frac{r_1 - r_2 + c}{2a}$$

$$\cos\beta = 1 - \frac{r_1 + r_2 - c}{2a}$$

These two angles appear in the Lambert equation for time of flight.

$$\Delta t = \left(\frac{1}{n}\right)\left[(\alpha - \sin\alpha) - (\beta - \sin\beta)\right]$$

This equation confirms Lambert's theorem. Although the semi-parameter can be calculated from the formulas for a and e, it can also be found by

$$p = \frac{2}{c}(s - r_1)(s - r_2)$$

where s is the semi-perimeter

$$s = \frac{r_1 + r_2 + c}{2}$$

which, by inspection, is equal to 2a ~ the major axis of the minimum energy solution.

This classical derivation is significant, in that it uses both foci ~ moreover, the very simple and elegant equations for time of flight have two functions - alpha and beta, the so-called Lambert angles.

Overall, these two important applications of the Two Body Problem reinforce the comments on geometric forces existing to impel the simple, direct solutions as they are posted here; enforcing a kind of symmetry that gives the empty focus and the center of the ellipse just enough influence to coordinate position and time of flight to be so intimately yet simply inter-related as has been illustrated.

VIII. Two New Forces

A powerful method of orbit determination is done using the f- and g-functions, which work on all conic sections. This theory applies a principle from linear algebra that if three vectors are coplanar ($\vec{r}, \vec{r}_0, \dot{\vec{r}}, \dot{\vec{r}}_0$ are all in the plane of the orbit) then one vector can be expressed as the sum of the two other vectors, if they are not collinear. Thus

$$\vec{r} = f\vec{r}_0 + g\dot{\vec{r}}_0$$

differentiating once,

$$\dot{\vec{r}} = \dot{f}\vec{r}_0 + \dot{g}\dot{\vec{r}}_0$$

A useful relation (i.e. the determinate of equations (3) and (4) as rows) is

$$\dot{g}f - g\dot{f} = 1$$

Which shows that $f, g, \dot{f}, \dot{g}$ are not independent. However, if any three are known then equation (5) can be used to solve for the fourth variable. The functions in terms of the orbital elements are as follows

$$f = 1 - \frac{a}{r_0}\left(1 - \cos\theta\right) \text{ where } \theta = E - E_0$$

$$g = \tau - \frac{\theta - \sin\theta}{n} \text{ where } \tau = t - t_0$$

$$\dot{f} = -\frac{na^2}{rr_0}\sin\theta$$

$$\dot{g} = 1 - \frac{a}{r}\left(1 - \cos\theta\right)$$

A variation of the equations can be used to find an earlier position (e.g. periapse if it's not known) from an existing position and velocity vector by

$$\vec{r}_0 = \dot{g}\vec{r} - g\dot{\vec{r}}$$

$$\dot{\vec{r}}_0 = -\dot{f}\vec{r} + f\dot{\vec{r}}$$

These two equations come from taking the inverse of the f and g functions.

Another interesting property of the f and g functions is that they are themselves (scalar) solutions to the two body equation, perhaps implying they have some fundamental significance.

$$\ddot{f} = -\frac{\mu}{r^3} f$$

$$\ddot{g} = -\frac{\mu}{r^3} g$$

The leading terms of series expansions in time of the f and g functions (i.e. those terms that are known) are useful when the initial position vector is known but not the initial velocity vector.

$$f = 1 - \frac{1}{2}\sigma\left(t - t_0\right)^2 + \frac{1}{2}\sigma\psi\left(t - t_0\right)^3 + ...$$

$$g = \left(t - t_0\right) - \frac{1}{6}\sigma\left(t - t_0\right)^3 + ...$$

where $\sigma = \frac{\mu}{r_0^3}$ and $\psi = \frac{\dot{r}_0}{r_0}$.

The f and g functions can be used to study the individual components of the position and velocity vectors.

$$x = fx_0 + g\dot{x}_0$$

$$y = fy_0 + g\dot{y}_0$$

$$\dot{x} = \dot{f}x_0 + \dot{g}\dot{x}_0$$

$$\dot{y} = \dot{f}y_0 + \dot{g}\dot{y}_0$$

where, $x_0\dot{x}_0 - \dot{x}_0 y_0 = h$

The f-and g-functions act independent of vector algebra (do the math);

$$\begin{vmatrix} f\vec{r} & g\dot{\vec{r}} \\ \dot{f}\vec{r} & \dot{g}\dot{\vec{r}} \end{vmatrix} = f\dot{g}\vec{r}\times\dot{\vec{r}} - \dot{f}g\vec{r}\times\dot{\vec{r}} = \left(f\dot{g} - \dot{f}g\right)\vec{r}\times\dot{\vec{r}} = \vec{r}\times\dot{\vec{r}}$$

These derivations qualify much of the geometry posed so far:

(1) $|h|$ is show to be as complex and sublime as suggested from the start

(2) f and g are solutions to the Two Body Equation - they are two scalars that satisfy it, as well as the position vector. This suggests some kind of new, disjointed forces taking place

(3) There are only two other symmetric points in an ellipse (other than the primary focus where the planet it situated) - the empty focus and the center of the ellipse - from which forces could logically originate

(4) These are not trivial solutions, the f- and g-equations; even though the eccentric anomaly is defined differently for parabolic and hyperbolic orbits. So they represent a fundamental and invariant relationship.

The f- and g-equations translate $\vec{r}_0$ and $\dot{\vec{r}}_0$ to any other point on an ellipse. The determinant of their matrix is equal to 1 - the same as for rotations. Consequently, the f- and g-equations are a rotation matrix that rotates elliptical coordinates ~ not circular coordinates, as is typical of rotation matrices.

The only practical way to rotate elliptical coordinates is to first translate to a conic section in which the path is a circle; rotate to a new angle; then translate back to the original orbital plane. This is essentially the process you go through using the mean and eccentric anomalies to advance a trajectory forward in time.

The mechanism to do all this, in orbital mechanics theory, is symbolic of a kind of in-plane rotation on circles of radius a and b, as has been suggested in previous sections.

The assumption has always been that this whole process hints of Relativity ~ that significant forces at Relativistic speeds exist, at least in the mathematics, at the orbital level for planets and satellites around planets. If the narrative thus far has not proven this to be true, it has at least shown reasonable evidence that it could be true.

If this were a civil suit, the weight of evidence would be in favor of my hypothesis ~ if only by a margin of 50.1% ~ which hopefully is enough for you to continue.

The iron clad proof will not come from pursuing this mathematical analysis of orbital theory any further. The requisite proof must come from hard numbers - data - and the logical place to look is the Solar System. Large bodies move in well known paths, and there is lots of data. The scale is many orders of magnitude large than the Two Body Problem, therefore the alleged forces should be evidenced in a more dramatic and quantifiable manner.

The problem is how to organize the data so that it lends itself to the kind of analysis we need. That is the objective if the next section. The remainder of this section is devoted to a rigorous analysis of the f- and g-equations. The idea is to try and establish some continuity to the logic established so far, and the evidence posed in this section is weak at best. Hopefully the following analysis - along with the model built in the last few sections - will dispel most doubts and provide enough gist for the mill to motivate continued exploration.

Structure of the F- and G- Equations

Begin with a heuristic look at Kepler's 2nd Law derivation.

$$\frac{dA}{dt} = ½r^2 \frac{d\theta}{dt} = ½r^2\dot{\theta} = \text{constant}$$

thus $$\frac{dA}{dt} = ½r^2 d\theta \text{ iff } r^2\dot{\theta} = \text{constant} = \left|\vec{h}\right|$$

The conservation of areal velocity is not limited to an inverse squared law force (e.g. planetary motion) but is the general result for all central force motion, which includes 1/r forces.

The slight error in the mathematics that is picked up by Relativistic calculations also implies that $\left|\vec{h}\right|$ is not constant in direction. Thus the plane of t he orbit has a slight wobble, just the kind of motion implied in the small eccentricity theory and exhibited at the origin of the symmetric plane. An electrical analog to the 2D harmonic oscillator can help to show how these new f- and g-forces interplay with the well known force of gravity. The f- and g-forces are all but nonexistent because, like a Whetstone Bridge, balance the system so that it behaves just like a local inertial reference system; or at least independent enough so that fractal theory exists, but uniquely.

Now consider the mathematics of this situation, by reproducing the derivation for the f- and g-equations, but instead of the generalized case for any value of eccentricity or orbital shape; consider them for e=.01 and p=1.

$$F = 1 - \frac{r}{p}(1-\cos\theta) = 1 - r + \cos\theta = (1-r) + \cos\theta$$

$$G_t = 1 = \frac{r_0}{p}(1-\cos\theta) = 1 - r_0 + \cos\theta = (1-r_0) + \cos\theta$$

$$FG_t = \cos^2\theta + \cos\theta(1-r+1-r_0) + (1-r)(1-r_0)$$

where the last term is $O(e^4)$ and can be neglected. The middle term cancels with a term developed later, both being something like

$e^2 * [\sin terms] * [misc]$ which are small and opposite in sign and $O(e^4)$

It was shown above that $FG_t = -GF_t$, so now calculate the second function and then compare the two sides.

$$G = \frac{rr_0}{\sqrt{\mu p}} \sin\theta = \frac{rr_0}{\sqrt{\mu}} \sin\theta$$

$$F_t = \frac{\sqrt{\mu}}{r_0 p}\left[\sigma_0(1-\cos\theta) - \sqrt{p}\sin\theta\right]$$

$$= \frac{\sqrt{\mu}}{r_0}\left[\frac{\bar{r}_0 \bullet \bar{v}_0}{r_0 p}(1-\cos\theta) - \sin\theta\right]$$

$$= \frac{\sqrt{\mu}}{r_0}\left[\frac{r_0 v_0 \sin\phi}{r_0}(1-\cos\theta) - \sin\theta\right]$$

Thus,

$$GF_t = r\sin\theta\left[v_0 \sin\phi(1-\cos\theta) - \sin\theta\right]$$

$$= -r\sin^2\theta + rv_0\sin\phi(1-\cos\theta)$$

where $v_0 = r_0\omega = r_0$ and $\sin\phi \cong \phi$ (small) and substituting $r = 1 + e^2\sin\theta + h.o.t.$ (higher order terms) you get

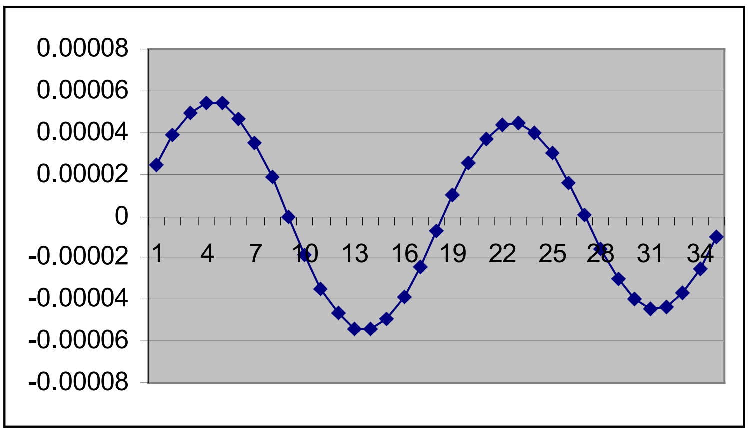
$$GF_t = -\sin^2\theta + e^2\phi\sin^2\theta + h.o.t.$$

Given the $\sin^2$ and $\cos^2$ terms and their sum = 1 for all theta, then $\sin^2 + \cos^2 + [terms] = 1$ for all theta, then the $[terms] \equiv 0$ for all theta.

Now to show the action of one set of terms, and the other set of terms must do the exact opposite, in order for the determinant to equal zero under all conditions. Neglecting the cos term and the $O(e^4)$ last term, consider the other term

$$FG_t \Rightarrow \cos\theta(1-r+1-r_0) \text{ recall } r = 1+e^2\sin\theta + h.o.t.$$
$$\Rightarrow \cos\theta(1-1-e^2\sin\theta+1-1-e^2\sin\theta_0)$$
$$\Rightarrow -e^2\cos\theta\sin\theta - e^2\cos\theta\sin\theta_0$$

which works out to be an elongated sine wave.



When we model the Solar System in the last few sections, this is just the kind of motion that happens at the origin - i.e. at the Sun - in a kind of Relativistic distortion of space near the Sun.

The model of the Solar System to be posed is a new plane with a distortion near the sun, a sinusoidal variation in the y-intercept of a line sweeping outward from the Sun. The origin is flexible, which is just what the above analysis shows.

The Complex Plane

A little solid math always helps to dispel doubts. So consider the Two Body Problem solution in the complex domain. Starting with the Two Body Equation itself,

$z = x + iy = \mathrm{Re}^{-it}$ where e^{it} is a counterclockwise rotation

$$\dot{z} = -i\,\mathrm{Re}^{-it} + \dot{R}e^{-it} = iz + \dot{R}e^{-it}$$

$$\ddot{z} = -i\dot{z} - i\dot{R}e^{-it} + \ddot{R}e^{-it} = -iz - i\left(\dot{z} + iz\right) + \ddot{R}e^{-it}$$

$$\ddot{z} = -2i\dot{z} + z + \ddot{R}e^{-it}$$

Notice how elegantly this representation breaks out the individual terms for the coriolis force, centrifugal force, and gravitational force. In Cartesian coordinates the only force was due to gravity. Again, we have a straightforward Two Body Problem magically projected into a rotating system. This suggests two actual forces exist in additional to the usual gravitational force.

Observe that a simple derivation in the complex plane produces the centrifugal and coriolis forces that are unique to rotating coordinates. In other words, the complex plane is identical to the rotating coordinate plane that is typical of the Three Body Problem. This will be used in Appendix I to advance the established theory discussed so far, into a new speculative model of Relativity.

IX Relativity Theory

TOPICS: Relativity's Three Proofs
The Invariant Plane
A New 1/r Force

The general trend in the history of astronomy has been from geocentric, to heliocentric, to barycentric (based on the center of mass). Now the f- and g-fictions seem to suggest another center of force. It's vague, just like Relativistic phenomena at Newtonian speeds; but nagging there nonetheless. It's a strong temptation to just out right pose a whole new coordinate system ~ but more justification is needed to do so.

Perhaps the solution will become more obvious if we try a little exercise in logic ~ to suppose a new heuristic/graphical solution that explains the three "tests" of Relativity better than Einstein's Relativity Theory. These three criteria used to "prove" Relativity are now considered one at a time, from the subtlest phenomena to the most obvious.

Time Dilation at High Velocities

With the advent of atomic clocks and the GPS system of satellites, it is possible to see an infinitesimal change in time in satellites orbiting at high speed. This phenomena was predicted by Relativity, successfully.

The problem with this sort of "proof" is that there are so many unknowns in the experiment itself. The radius of Earth is found statistically, using the hugely complex gravity model and satellite measurements. Even a tiny error in the origin of the Earth centered reference frame used in the Relativity calculations would offset any Relativistic affects. In fact, the gravitational models of Earth not only include Earth's "nominal" radius as a term in an infinite series expansion, but a Relativistic term as well. As the model becomes more accurate, both terms change. The Relativistic term could very well go to zero, eventually.

That's not a very strong proof that Relativity Theory is correct. If there were no other experiments that evidenced relativistic affects, this time dilation one wouldn't pass anybody's muster. A much simpler explanation - and correction - would be a simple adjustment to the inertial reference frame origin; compare that to the pages long equations of time and space distortion of Relativity Theory, and nobody in their right mind would pay heed to Relativity.

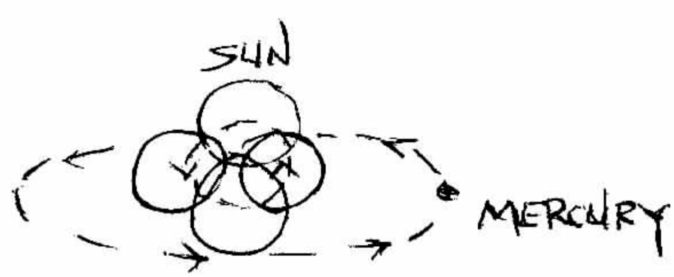
Precession of Mercury's Orbit

The equatorial bulge of Earth causes orbits of Earth satellites to precess over time ~ the line of nodes rotates, all other aspects of the orbit being unchanged. Mercury's orbit around the sun also precesses. Until Relativity theory, everybody assumed - because nothing at all was known about the mass distribution within the Sun - that Mercury's orbit precessed because of a mass anomaly within the Sun analogous to Earth's equatorial bulge.

We still know absolutely nothing about the distribution of mass inside the Sun. Dynamic models of a rotating mass, in fact, require an equatorial bulge like happened for Earth, and like what exist for all the other planets ~ including the gas giants - Jupiter, Saturn, Neptune, and Uranus. If the outer planets all have equatorial bulges, then the odds are that so too does the Sun.

Given the simplicity of the Newtonian explanation (and it is still taught by many astronomers and physicists) and the extreme complexity of Relativity calculations, it's irrational to choose Relativity.

Actually, it doesn't even take an actual mass anomaly to explain Mercury's precessing orbit. A small error in the inertial reference frame away from Sun's center could make Mercury see a bulge that is simulated by the Sun's own small orbit because parts of the Sun's mass - especially at the equator - are exposed to Mercury for longer than is typical for a straight orbit.



The official barycenter of the Solar System - which is essentially the center of mass of the Sun-Jupiter system - moves at the rate of Jupiter's orbit (12 years/revolution), which means the Sun also makes a 12 year orbit. This is a long term perturbation, if that; and insufficient to precess Mercury's orbit.

In the absence of any data on the distribution of mass within the Sun; with the previous model of Jupiter explaining the Great Red Spot by a mass anomaly within that gas giant (and Neptune's Dark Spot); and with the other rotating gas giants in the Solar System all having equatorial bulges; the evidence is strong that Mercury's orbit is caused by just that, a simple mass anomaly within the Sun. There certainly isn't enough evidence to justify Relativity.

Bending of Light by the Sun

If you calculate the deflection of a photon of light (all mass) by the Sun's gravity according to Two Body Theory, the measured deflection is exactly half that which is observed experimentally. This assumes that the photon is 100% mass, and no charge. You have already seen that charged particles exhibit the same hyperbolic deflection as gravity. There is no proof that says part of the deflection may be due to charge and part to gravity.

If you assume the Sun's mass is uniform, and that an equatorial bulge is simulated by the Sun's motion in a small orbit around some point to cause Mercury's orbit to precess; this supposition of a different center of mass suggests the possibility of a 1/r force originating at the new mass center. (Recall that a 1/r force at the center of an ellipse, not at a focus, causes elliptical motion just like in the Two Body Problem.) This could double the deflection, to match the experimental value.

You might say this 1/r force is an "equal and opposite reaction" to the Solar System as a whole. If you have a beach ball on a table, Newton's Third Law says the table exerts an equal and opposite force on the ball to counter balance the ball's weight on the table. The Sun can't just float there in space, the planets tethered to it; there must be some reaction that keep the Sun where it is.

So now Relativity violates Newton's Third Law ~ which means the evidence supporting Relativity must be extremely compelling. Moreover, every instance where Relativity is proven must also show that Newton's Third Law is violated. That is not the case for either of the first two experiments, weakening the case for Relativity substantially. So, the whole weight of proof for Relativity lies with the observed bending of light by the Sun. This increases the leeway for consideration of a well posed alternative theory.

If you have a 1/r force you must also limit it's "range" by establishing an event horizon, like what exists around Black Holes as predicted by Relativity. In recent years there has been speculation, by Stephen Hawking and others, that there are gradations of Black Holes ~ with the kind of 1/r step-force proposed here. So, here's an alternative theory to Relativity in three simple statements:

I. A 1/r force at a "virtual center" located at a precise inertial reference frame origin has an "event horizon"

II. For small bodies like planets the 1/r force is not active but is rather evidenced as a slight shift in the inertial reference frame, as if the force itself is pent establishing this inertial origin

III. For stars the 1/r force is approximately equal to the star's gravitational force, but has an event horizon at the corona, beyond which only gravity can be perceived

This explains away all Relativistic phenomena, fits in nicely with the latest thinking on Black Hole mechanics, has a regular old Newtonian explanation, and even suggests a dynamic mechanism to keeping the Earth's core molten after billions of years, and the Sun's fires burning.

That is the model. The rest of the material will be used to quantify this model from data in the same kind of inverse problem solved by Kepler, but this time with more accurate data and 100 years of additional understanding of gravity and the cosmos.

The Invariant Plane

The formal technical paper in Appendix I, "Orbital Motion in the Frequency Domain" shows a statistical model of the Solar System, developed by taking an average least squares fit of the orbits of all the planets (positions over time, disregarding mass) ~ the actual method used is described at length in the paper. There are several important characteristics of this model:

(1) A line from the Sun outward through the mean position of the planets was calculated - the slope of this line varies as a single sine wave, as does the y-intercept, as the line sweeps out a 360° revolution around the Sun

(2) The uniform variation in slope identifies a flat plane beyond the immediate vicinity of the Sun

(3) The slight variation in y-intercept creates a small depression or event horizon at the origin, near the Sun's center of mass

(4) All planetary motion with respect to this new plane because of eccentricity and inclination is exactly uniform ~ the new "symmetric plane" precisely bisects each planet's orbit

(5) A mechanical model of this plane consists of a straight line affixed to the circumference of two rotating circles, one near the origin and one in the vicinity of the asteroid belts

The mechanical model or sub structure of this new "symmetric plane" is exactly what you might expect to be the action of the f- and g-equations ~ translating motion, by virtue of two sinusoidal motions, to a new coordinate system in which motion is uniformly sinusoidal - and, thus, predictable. This model has the event horizon as well as a new origin or virtual center of the inertial reference frame. Consequently, the new coordinate system by itself satisfies two of the problems "explained" by Relativity ~ add a 1/r force within the event horizon, and Relativity is fully explained.

The 1/r Force Independently

The geometry of this new symmetric plane being what it is, shows the existence of a new frame of reference ~ outdating conventional reference frames by just enough that the difference is modeled by Relativity. The three most obvious consequences of this new reference frame are as previously noted.

The mathematical process used to develop this new frame of reference - i.e. the Fourier and Laplace Transforms - validate the model, and the reasonable explanations it produce for many heretofore unexplained phenomena, further authenticate the model. (Complete details are in the Appendix papers.)

In the absence of further analysis, this model for Relativity is exactly as Copernicus' heliocentric model was before Kepler's and Newton's Laws ~ a simple, elegant model in stark contrast to the clumsy and complex geocentric model of Ptolemy.

The problem is to establish validity. Since Relativity has been applied successfully on both the atomic and astronomical levels, a competing body of work must be a "theory of everything." Or, you must provide some independent proof of a 1/r force. The latter is provided in Appendix II in the paper, "A New 1/r Force in Orbital Motion."

Appendix

This Appendix has two purposes. First, to carry forward the Relativity model thus far developed from existing science and theory. Second, to test the new theory itself.

A major criteria for evaluating a new theory is that it must explain things not explained by any other theory. That is, in addition to being simpler, and to being inclusive of both Newtonian and Relativistic physics, a new theory must go farther than either in advancing our understanding of astronomy ~ which, as you will see, it does to a certain extent.

The theory I have created goes further than that. It advances our understanding of basic mathematical operations like Fourier and Laplace Transforms. It gives a new meaning and depth to simple harmonic motion, which is ubiquitous to both mechanical and electrical engineering.

Finally, in posting two new forces in simple harmonic motion, the theory implies that existing machines and electrical devices can be made more energy efficient and long lasting by eliminating these spurious forces. In this regard, the test of the theory will be easy enough to do with conventional machines, and not have to wait for some obscure astronomical experiments.

Appendix I

Orbital Motion in the Frequency Domain

By WH Clark, MSE

ABSTRACT

A powerful approach in Celestial Mechanics is to visualize orbital motion in a rotating coordinate system. This paper translates motion into rotating coordinates by applying the Fourier Transform to the motion of planets in the Solar System. The resulting analysis in the frequency domain suggests the planets, in this Ten Body Problem, are in motion about points that are the equivalent of Lagrange Points in the Three Body Problem. This hypothesis is tested by taking a Laplace Transform of planetary motion, which produces dramatic results - with new insights into the nature of gravity, planetary motion, and the complex interaction between planets in the mutual perturbation of each others' orbits.

Text

The Fourier transform of some function f(x) is defined by

$$\hat{f} = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}dx \qquad (1)$$

The rights side of this equation is an inner product between the two functions f(x) and $e^{-i\omega x}$. From analytic geometry, the exponential function is represented in the complex domain by a unit vector rotating counterclockwise at ω rad/sec to sweep the plane

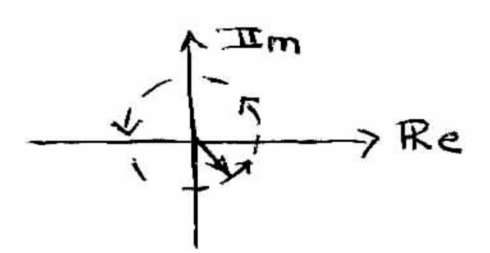


Figure 1 ~ $e^{+i\omega t}$

We normally think of energy in the time domain. Fourier Transforms are an alternate view of the distribution of energy, in the frequency domain. To see how this changes the perception of orbital motion, assume the celestial plane of the Solar System is the complex plane. A planet moving around the Sun on the unit circle is Figure 1, in this context.

If this were a low earth orbit, the situation is complicated by drag so that eventually the orbit deteriorates at a constant rate.

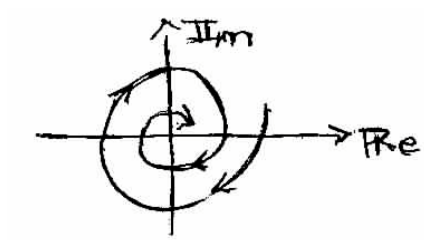


Figure 2 ~ $\left(e^{-j\omega t}\right)e^{-t}$

Back to the situation in Figure 1, a planet in a slightly eccentric orbit can be approximated (assume ω = 1) by

$$\left(e^{-j\omega t}\right)\left(\sin t\right) \qquad (3)$$

Similarly, a function orthogonal to the celestial plane approximates a planet orbiting at a small inclination.

Now consider a non rotating system.

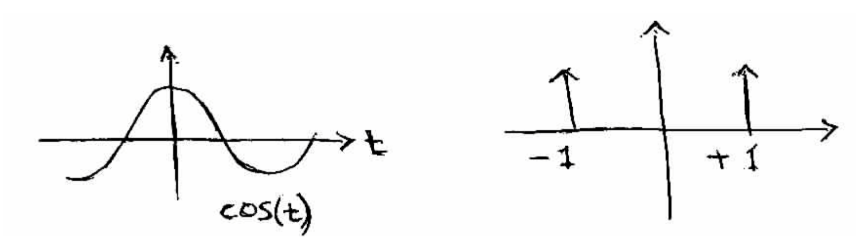


Figure 3

This implies that an orbit with small eccentricity is a "delta" function in the frequency domain, similarly for an orbit at a small inclination to the celestial plane.

Small Eccentricity Orbits

Consider a unit circle inscribed within an ellipse of small eccentricity, with the center of the circle coinciding with a focus of the ellipse. It is easy to see that the variation between the radii varies systematically, if not exactly sinusoidally. In practice, an ellipse of eccentricity .01 varies from the unit circle by a cosine wave of magnitude .01

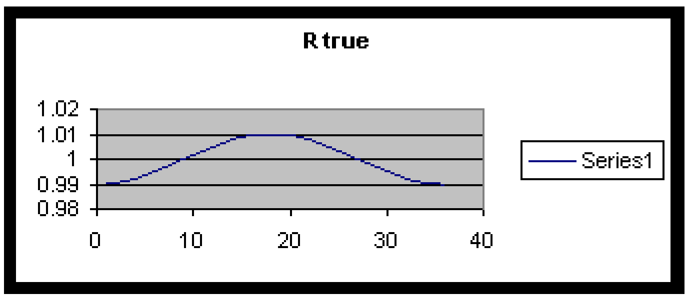


Figure 4 An ellipse of eccentricity .01 versus the unit circle

The maximum error can be modeled accurately by another cosine curve of half the wavelength, and a magnitude of .00005 And so forth.

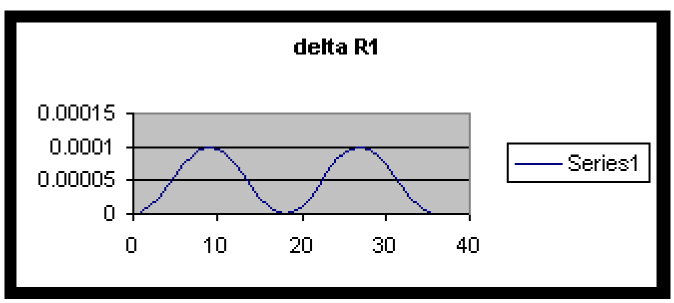


Figure 5 An ellipse versus the first term approximation

This is the basis for Fourier Series: any curve can be reproduced exactly by the sum of sine and cosine waves. This is shown above for the case of a small eccentricity ellipse.

As the eccentricity of an orbit increases, it takes more sine/cosine functions to approximate it. Nevertheless, all the energy is at a fixed number of frequencies, and the higher the frequency the lower the amplitude.

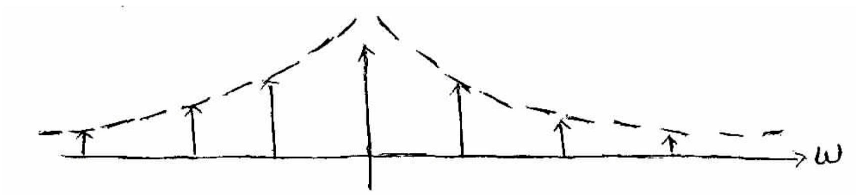


Figure 6 Quantized Energy Levels

Arguably, the smaller spikes for the planets in the Solar System are perturbations from other planets. If you order the planets by eccentricity and consider the relative perturbations on each, you get the following

Planet	**Eccentricity**	**# perturbations**	**comments**
Venus	.006	E	Small/big <<< 1
Earth	.006	Mars	Small/big << 1
Neptune	.008	U	Big/big ~ 1
Jupiter	.05	U,S,N	3-
Uranus	.05	J,S,N	3
Saturn	.06	J,U,N	3+
Mars	.093	J,S,U,N	Big/small ~ 5
Mercury	.20	S,E	Big/small ~ 6
Pluto	.25	J,S,U,N	Big/small ~ 7

The comments refer to the relative size of the planet versus the perturbing bodies.

The efficacy of the symmetric plane is evident if you consider that the periapse of all the planets are spread out around 360 degrees. It's amazing that such a random system could exhibit such sublime symmetry. If you order the planets by eccentricity, and consider the influence on each planet you get the eccentricity as noted; but also the inclination of the orbit is by the figure fixed at a maximum by the orbit of Jupiter - i.e. the limits of the Solar System's angular momentum plane, usually called the invariant plane. This explains, in a very simple way, the orbital elements of all the planets.

A Symmetric Plane

The new reference plane we seek is one in which each planet has the same relationship as eccentricity is to the unit circle ~ this plane must, consequently, have the minimum orbital inclination collectively. Figuratively speaking, it must be a least squares fit to the orbits of all the planets. This fit has to be independent of mass.

A powerful technique, frequently applied in Celestial Mechanics to gain insights into an otherwise intractable problem, is to transfer coordinates to another frame of reference. What is sought here is to find a three dimensional plane in space about which the motion of all the planets in our Solar System is perfectly symmetric.

Consider the orbital positions of the planets, reduced to a manageable set of data. The method in concept is as follows:

(1) The position of each planet at 0° heliocentric longitude is calculated from the orbital elements

(2) A linear least squares fit is made of these nine data points, giving a slope and intercept

(3) This is repeated at 10° increments through 360° of longitude

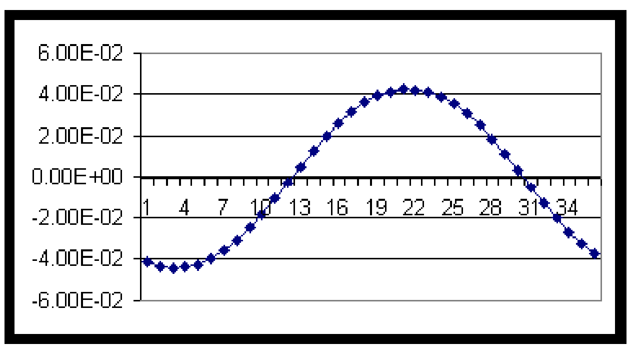


Figure 7 Slope vs. Heliocentric Longitude

The intercept of the 36 lines varies sinusoidally. This creates a distortion at the origin; i.e. in the vicinity of the Sun.

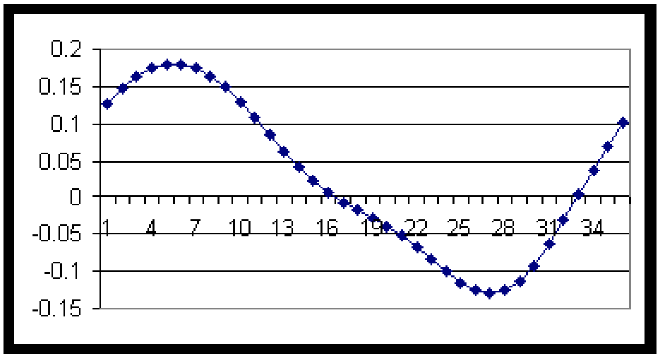


Figure 8 Intercept vs. Heliocentric Longitude

The true measure of this new plane in space is how symmetric the motion of the planets is with respect to it. This motion is shown in the topmost graph to be uniformly sinusoidal in the z-direction.

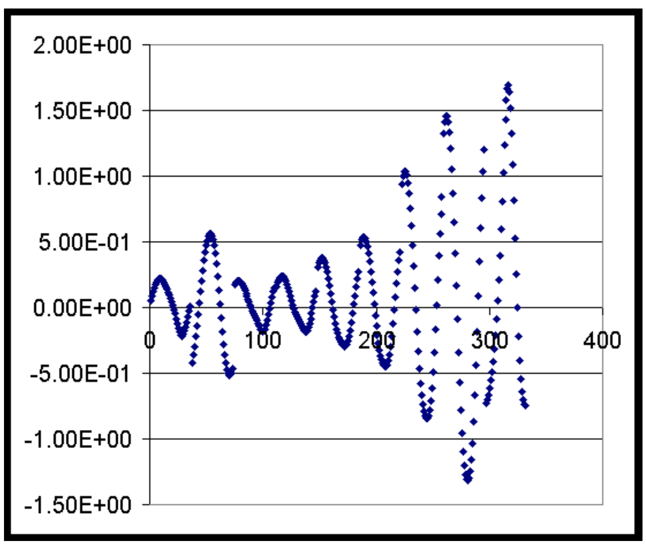


Figure 9 Position Perpendicular to the Symmetric Hyperplane

Observe that all planets (excepting Venus, and Pluto) are sinusoids exactly in phase, with the same leading phase angle. Venus and Pluto are exactly pi radians out of phase with the other planets; they also exhibit retrograde rotation, so being out of phase with the other planets in prograde rotation is a logical consequence. (The graphs of the planets are arranged in their actual sequence but adjacent to each other, disregarding the large gaps between each orbit.) The curve for each planet is not a closed curve because of the action at the origin of this new coordinate plane, which is not fixed at barycenter, which results in one cycle of a sinusoid, whereas a coordinate system fixed at barycenter would generate an elliptical shaped pattern versus the symmetric plane.

The important thing is that the motion of each planet is symmetric with respect to this plane. In the course of a revolution around the Sun, each planet actually traces out an ellipse in an inertial coordinate system. The geometry of the ellipse determines the inclination and eccentricity of the orbit, and the orientation of the ellipse determines the longitude of the ascending node. In a manner of speaking, there is two body motion in a local coordinate system rotating with each individual planet.

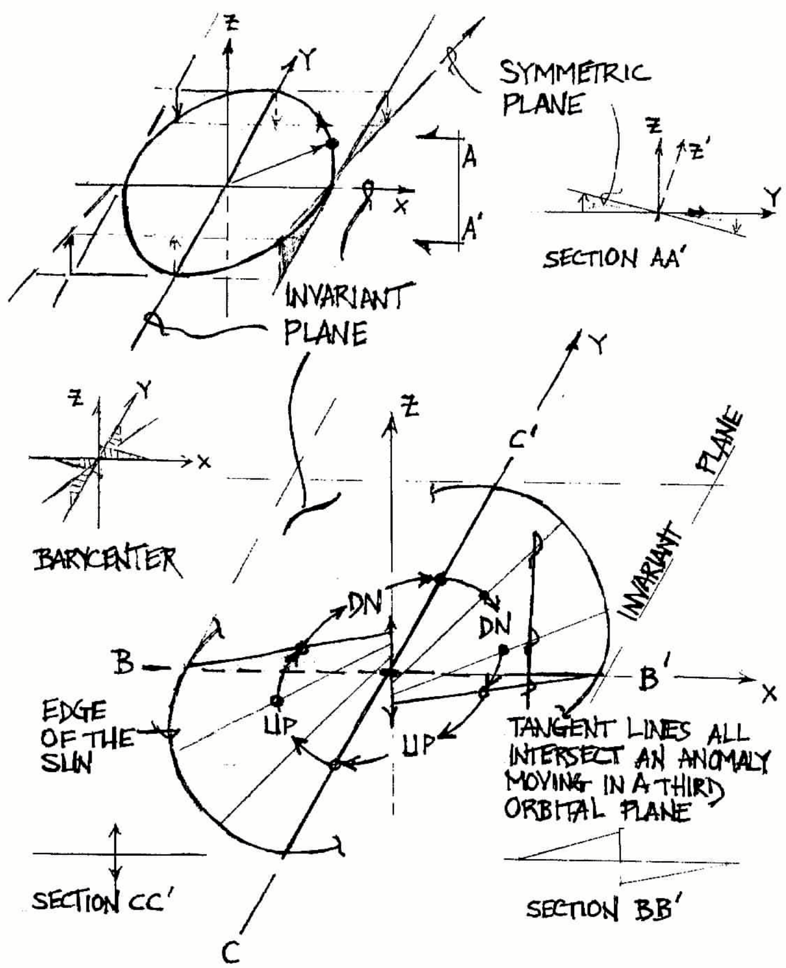
Upon closer inspection, the symmetric plane derived here is not the sort of static plane that is usually associated with the word – that is, it is not completely flat in 3D space. There is a relativistic distortion of space at the origin. (See the drawing on the following page.)

The symmetric plane is not so easy to conceptualize, but consider a straight line affixed in sliding contacts to two circles: a small one with center at the Sun and a larger one with center in the vicinity of Jupiter's orbit, near to the transition between the small inner planets and the much, much larger outer planets. This line, rotating through 360º in 3D space forms the symmetric plane; both circles also rotating through 360º as the mechanical system makes one revolution around the Sun.

Stability Points

This analysis has been achieved using an algorithm that approximates orbits of the planets in inclination and eccentricity using the first term in the Fourier series approximation.

The important thing to notice is the similarity between all the planets in Figure 9 ~ they all have similar shaped curves (the start and end are different because of the y-intercept in the coordinate system used),



differing only between the prograde and retrograde planets, which supports the idea of the outlying spikes being perturbations traceable to specific planets. This remarkable consistency shows a common, direct relationship between perturbations (spikes) causing eccentricity and inclination ~ orthogonal vectors coincident to the same point on the axis.

Fourier transforms often reveal a strong periodic component that is not noticeable in the time domain. That has certainly been the case in this study. The eccentricity and inclination of orbits are directly and orthogonally related in the non-rotating frequency domain. This overall symmetry implies each planet acts upon the others from its own "center of power."

In the rotating coordinates of the Three Body Problem, this "center of power" for an object in a stable, elliptical orbit is a "Lagrange Point." The same exists in the Ten Body Problem of the Solar System ~ combine the orthogonal (coincident) vectors for the eccentricity and inclination of any given planet, and you have a single vector rotating in some out of plane orbit that describes the orbit of a planet about its "Lagrange Point." The planet is in a stable orbit about this point on the symmetric (rotating) plane, and this same "point" in the frequency domain is the source of perturbations upon the other planets. All together, this intricate interconnectedness gives the whole system a long term stability ~ i.e. truncating the otherwise infinite series of spikes that are implied by Figure 6.

A troubling feature of this analysis is the scale difference between the inner and outer planets (neglected in the symmetric plane plot in the direction radially outward from the sun.) The solution in analytical mathematics to such a problem is to multiply everything by a negative exponential and to present the whole system as a Laplace Transform.

Given the Solar System is in three dimensions (plus time), the aspect upon which to apply this negative exponential is not obvious. A clever (perhaps devious) solution is to suppose a second rotating exponential orthogonal to the Fourier exponential. This cannot be done analytically, but geometrically the effect is to generate a wavefront originating at barycenter, with a specific wavelength and amplitude. The characteristics of this wave are as a least squares fit, one that most closely fits the motion of the planets with respect to the symmetric plane.

Heuristically, this new wave pattern is the mechanism by which the sun exerts its influence upon the planets, as a kind of fancy gravity wave. The following trial-and-error approximation assumes this new iteration of gravity is related to the mechanism by which the axis of rotation of each planet remains fixed in its orientation to the same point in the celestial sphere throughout its revolution around the Sun.

The System Wave

The first objective of the analysis is to create a more stable system by better explaining systematic deviations from the symmetric plane. In order to further define the motions, first introduce another variable: the planet's axis of rotation. This axis is known to always point to the same spot in the celestial sphere, throughout its revolution around the sun. During this movement, the planet moves evenly above and below the ecliptic plane. These characteristics can be conveniently represented if the planet is considered to be associated with a longitudinal wave. If the axis of rotation is always tied to this sinusoidal wave, then as the planet moves about the sun the elliptical shape of the orbit causes the planet to move closer and farther away from the sun. If this motion happens on a sinusoidal wave, the axis will always point in the same direction.

Otherwise, the choice of a sinusoidal waveform explains the three movements that occur simultaneously with respect to the new system plane: the elliptical shape of the orbit, the movement above and below the system plane, and the axial inclination of the planets pointing to a fixed spot in the heliocentric sphere while the other motions occur.
The data used in the analysis is listed in Table 1. The semimajor axis is an approximation of the planets' average orbital distance from the sun. A value "X" is the range of motion of the planet in its orbital plane.

Planet	*X*	Axial inclination	Semi major axis
Mercury	5.8 E7	0.0 degrees	1.2 E7 km
Venus	1.1 E8	R179	6.7 E6
Earth	1.5 E8	23.5	2.5 E6
Mars	2.3 E8	25.2	2.1 E7
Jupiter	7.8 E8	3.2	3.8 E7
Saturn	1.4 E9	26.8	7.8 E7
Uranus	2.9 E9	R98	13.2 E7
Neptune	4.5 E9	29.0	3.7 E7
Pluto	5.9 E9	90.0	1.5 E9

Table 1

The next task is to find the wavelength of a common "system wave." After trying several solutions, it was evident that the outer planets would be the controlling elements, so the search focused on the lowest common denominator of their orbital range. This value, 23.3 E6 km also had to fit the entire range of movement of the inter planets.

The second column of Table 2 lists the aspect of each planet upon this wave – the remainder of division by the proposed wavelength. This value was used to situate each planet at the proper location upon the wave. Note that positions are only approximate in this two dimensional system: the wave is actually a spiral or helix in three dimensions. A possible presentation of the "system wave" is a family of curves called the Cochoids of Nichomedes, with the parametric equations:

$$x = a + \cos t \text{ AND } y = a \tan t + \sin t \qquad (4)$$

These generate a two-part curve: a loop similar to a prolate cycloid.

The second objective of the analysis was to further develop the study to define the uniform deviations from the new system plane, so that a more stable system could be devised. This has been done for the original positions by simply introducing another variable and trying to match it. By seeking a close fit to this third variable, axial inclination, the affect was to more closely resolve the previously analyzed parameters. This affect becomes more pronounced as the analysis continues in greater detail.

Planet	*Division by 23.3 E6*	*Inclination*	*from the ecliptic*
Mercury	2 + 1.1 E7 km	7.0 deg	7.0 E6 km
Venus	4 + 1.7 E7	3.4	6.5 E6
Earth	6 + 1.0 E7	0.0	- -
Mars	9 + 2.0 E7	1.9	7.6 E6
Apollo Group	*9 + 7.8 E6*	-	-
Belt Asteroids	*16 + 2.2 E6*	-	-
Trojan Group	*33 + 3.6 E6*	-	-
Jupiter	33 + 1.1 E6	1.3	17.0 E6
Saturn	60 + 2.0 E6	2.5	61.0 E6
Uranus	124 + 1.0 E7	0.7	35.0 E6
Neptune	193 + 3.1 E6	1.8	141 E6
Pluto	253 + 5.1 E6	17	172 E6

Table 2

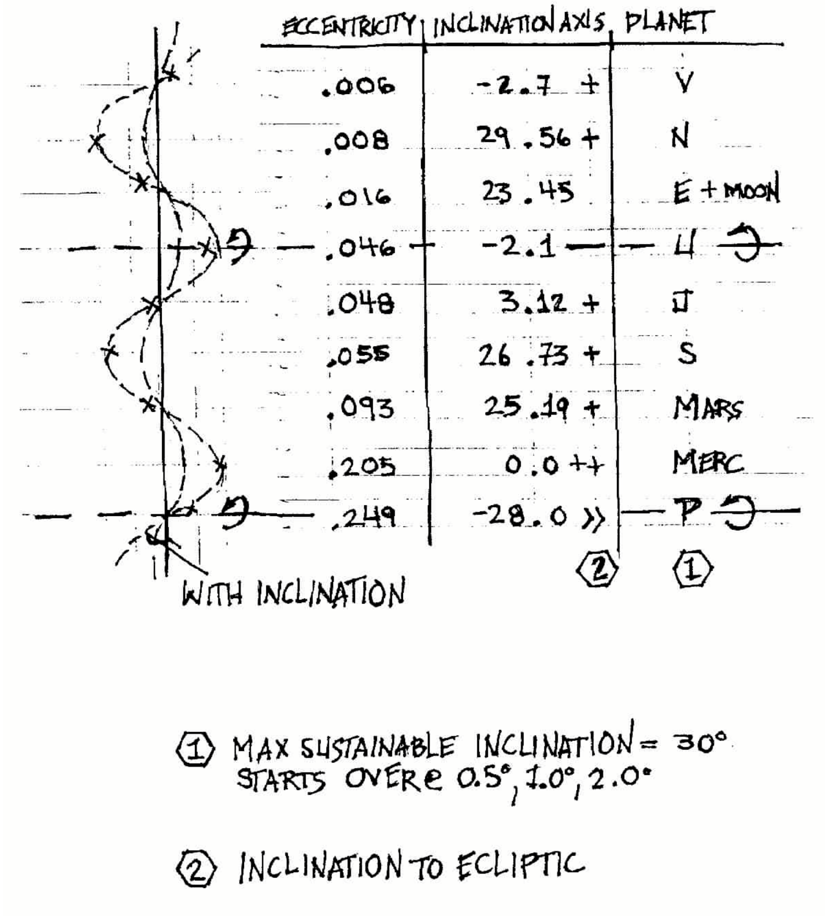


Figure 10 is a sketch of the planets situated upon either of the two parts of the cycloid, according to the average orbital distances in Table 1. The axis is shown as an arrow pointing according to the right hand rule for the planets' rotation. The positions of the various planets on the waveforms can now be fit to the wave.

First, the motion of Mars moves uniformly on either side of an inflection point in the wave, its axis inclined at about the right amount. Jupiter does the same thing on the lower wave, which arches less to correspond with the lesser axial inclination of Jupiter's axis. Uranus, also on the lower wave, moves through an entire wavelength, with an eight degree axial inclination off the vertical, slightly greater than Jupiter – since it traverses more of the wave.

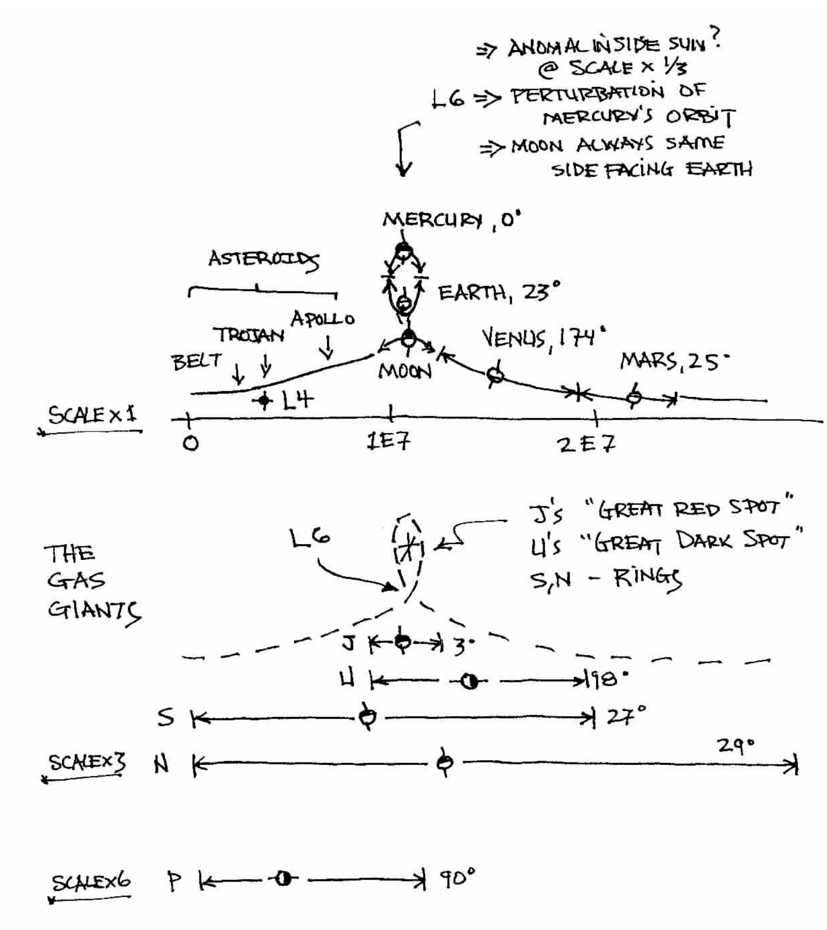


Figure 10 The System Wave

Next, Saturn on the lower wave traverses two wavelengths, with a much greater axial inclination. The center point of the wave can be thought of as an inflection point, motivating the body to rotate in order to maintain rotational stability – a two dimensional vista of the singularity at the sun discussed earlier. It is interesting to note that all four planets (with the possible exception of Pluto) have almost the same rotational period –

about ten hours. This would imply some similarity in the wave analogy, and it is there in the form of the inflection of the wave traveled.

Back to the upper waveform, there is Mercury shown atop the loop, It moves back and forth on the waveform the distance of its ellipticity, maintaining its axis pointed in the same direction. Although the planet is known to have an axial inclination of zero, the wave may be inflected so that the pattern traced is less exaggerated. The geometry of that area of the plane fits this expectation.

The Earth-Moon system is the most striking aspect of the system, buttoning up the sides of the loop. They are unique in the solar system, the Moon being the largest satellite with respect to its host in the whole solar system. This may be a necessary prerequisite for matter to exist in dynamic stability at the cusp of the wave front as shown. In which case Earth occupies the upper portion of the loop, the moon the lower, in its own orbit. The moon's orbital plane is inclined at five degrees to the Ecliptic, as is indicated by the inflection of the lines (versus the much higher inflection of the upper cusp, indicative of Earth's 23° axial inclination).

The close symmetry of the least squares fit to a sinusoidal slope and a sinusoidal motion of the y-intercept, and the uniform displacement of the midline of the sinusoidal motion of the planets about this plane, then a two-part pattern such as that envisioned most closely realizes both conclusions. In affect, the defining plane of motion then becomes the bottom half of the above parametric equations wave for the inner plane.

Planet	Movement w.r.t. the system plane (Au)	Center line	Radius
Mercury	-0.30 + 0.21 = 0.51	-0.05	0.26
Venus	-0.27 + 0.17 = 0.44	-0.05	0.22
Earth	-0.19 + 0.17 = 0.26	-0.06	0.13
Mars	-0.19 + 0.09 = 0.28	-0.05	0.14
Jupiter	-0.10 + 0.02 = 0.12	-0.04	0.06
Saturn	-0.30 + 0.17 = 0.47	-0.06	0.23
Uranus	-0.77 + 0.74 = 1.51	-0.03	0.75
Neptune	-1.21 + 1.21 = 2.42	0.00	1.21
Pluto	-1.10 + 1.60 = 1.70	-0.25	0.85

Table 3

Another way to look at the "system wave" is as an envelope curve or "envelope surface" made up of segments of all nine planets' orbital configurations. Thus the analysis here is more than a geometric curiosity but a common aspect of complex dynamical systems.

It is also important to note that, if nothing else, the "system wave" shows an exact correlation between orbit eccentricity and inclination or the orbit (also, perhaps, inclination of the axis of rotation) as was surmised in modeling the low eccentricity ellipse, and vice-versa. Presumably perturbations to the 3BP model by other planets would cause rotation on the axis, as was suggested here.

Conclusion

Note that the origin of the "system wave" is at the sun, not at barycenter. That explains why the plot of the orbits versus the symmetric plane is skewed ~ sinusoidal, and not elliptical. The origin has a Relativistic distortion, as implied by Relativity Theory.

Other solutions were tried, but the one that works best is a helical wave, as indicated. This helical wavefront propagating outward from the sun along the axis of the new symmetric plane eliminates the first term (spike) of the Fourier Transform approximation of the planets. This is to be expected, because it represents the sun's gravitational influence, leaving only inter-planetary perturbations (each of which presumably acts just like the sun's)

Appendix II

A New 1/r Force in Orbital Motion

by WH Clark, MSE

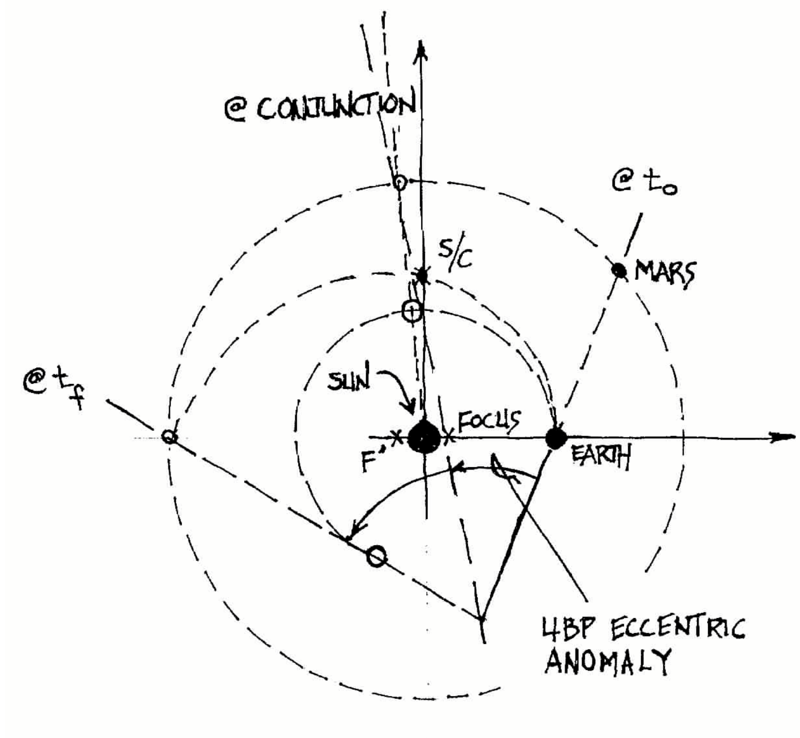
ABSTRACT

A simple graphical study of the Earth to Mars trajectory shows how a new 1/r force coordinates the motion of the two bodies, and suggests that the careful design of a spacecraft flight path could take advantage of this new force. This has been shown to be the case, in fact, using a computer model of the Mars flight path. Having validated this new force experimentally and conceptually, its place in the lexicon of modern physics is suggested to be the key to a Unified Field Theory. As such, replacing Relativity as a gravitational phenomena of electromagnetic forces, with a brand new Newtonian force that has its origin at the nucleus of the inertial coordinate system, conceivable allows a consistent representation of all forces in a single context. This paper is limited to general concepts, leaving the formal analysis to others.

The Earth to Mars Trajectory

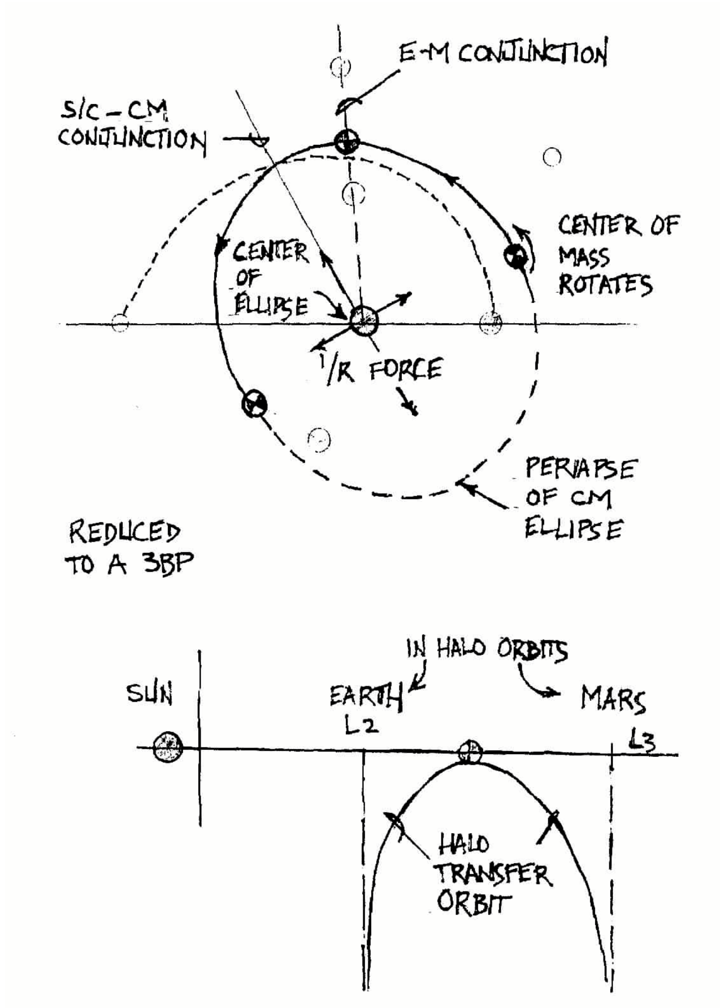
Consider the following illustration of the interplanetary trajectory from Earth to Mars. The spacecraft (s/c) leaves Earth as indicated, and arrives at Mars pi radians later. The initial and final positions of Earth and Mars during this period are as shown. The line labeled conjunction is the point at which the Sun, Earth, and Mars are on a straight line; at which time the s/c is a little to the left near the y-axis. Drawing a straight line through the initial and final positions of Earth and Mars shows that there exists a remote point about which the motion of Earth and Mars is consistent - both bodies remaining on this line as it sweeps from right to left; and, in fact, the s/c remains on this line as well - i.e. it's motion is almost twice as fast near Earth as at Mars, as corresponds to this diagram.

What this simple diagram has done is to reduce a Four Body Problem to a Three Body Problem. That is, referring to the second illustration, Earth and Mars can be replaced by a single body whose motion follows the path indicated - i.e. the Center of Mass of the Earth-Mars bodies. Analytically, this eliminates six unknowns from the problem and makes it as easy to solve as the Three Body Problem.



Observe that the closed form solution puts the Center of Mass of the two planets in an elliptical orbit, meaning the solution is periodic and stable. The Sun or central body is not at a focus of the motion as is typical with inverse squared forces like gravity; rather the central body is at the center of the ellipse, which is characteristic of inverse or 1/r forces. This is a simple problem in the analytic geometry to prove; i.e. uniform motion under the influence of a 1/r force is elliptical with the central body at the center of the ellipse.

In this new Three Body presentation, the Earth and Mars are at the colinear points L2 and L3 and the spacecraft is in a transitional orbit as indicated. The 1/r force can be thought of as the influence in the rotating 3BP that keeps Earth and Mars in this configuration.



CONCLUSION

This 1/r force signifies the beginning of a new fractal level, dividing the inner planets and the outer gas giant planets. The anomaly associated with the system represents the action of a force, just as the f- and g-forces were associated with the mean and eccentric anomalies of the Two Body Problem. It is easy to see that the origin of the 4BP anomaly is at Earth's moon, which is in the middle of the cusp of the System Wave. This projects the geometry into 4D or the complex domain.